Structural Models of the Firm under State-Dependent Volatility and Jump Process Asset Dynamics

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Abstract

We generalize the asset dynamics assumptions of Leland (1994b) and Leland and Toft (1996) to a much richer class of models. By assuming a stationary corporate debt structure with constant principal, coupon payment and average maturity through continuous retirement and refinancing as long as the firm remains solvent, we obtain analytical solutions with the state dependent diffusion volatility following the constant elasticity of variance (CEV) process for the variables of interest, including corporate debt value, total levered firm value and equity value. We also develop an efficient numerical algorithm with mixed jump diffusion asset dynamics, by adopting a restricted structure of default times, and derive numerical solutions for several variables of interest. We study the impact of state dependent volatility and jumps on the optimal capital structure, the debt capacity, the term structure of credit spread, the duration and convexity of risky debt, the equity volatility, the asset substitution impacts and the cumulative default probabilities (CDP). We also find that the term structure of CDP generated by the CEV diffusion process with a negative relationship between asset value and asset volatility could explain the CDP puzzle of structural models for short term debt by examining Moody's historical CDP data.

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1. Introduction

A very large number of studies, both theoretical and empirical, on corporate bond pricing and the risk structure of interest rates have appeared in the literature following the pioneering work of Merton (1974) and Black and Cox (1976), which in turn were inspired by the seminal Black and Scholes (1973) model of option pricing. These studies adopted the methodological approach of contingent claims valuation in continuous time, in which the value of a firm's assets played the role of the claim's underlying asset and allowed the valuation of the various components of the balance sheet under a variety of assumptions. This approach has been shown to be sufficiently flexible to tackle some of the most important problems in corporate finance, such as capital structure, bond valuation and default risk, under a variety of assumptions about the type of bonds included in the firm's liabilities. The resulting models came to be known as *structural models* of bond pricing, as distinct from another class of models known as reduced form models, in which there is no link between the bonds of a given risk class and the firm's capital structure.[3]

Under continuous coupon payment and first-passage default[4] assumptions, Leland (L, 1994a, b) and Leland and Toft (LT, 1996) first studied corporate debt valuation and optimal capital structure with endogenous default boundary for infinite maturity debt and finite maturity debt, respectively. Because of the computational complexity of the valuation expressions, a major emphasis in the structural models was placed on the derivation of closed form expressions, rather than numerical results based on approximations[5] or simulations.[6] Such a focus allowed relatively easy estimations of numerical values given the parameters of the model, but at the cost of maintaining simple formulations of the mathematical structure of the asset value dynamics, in which a univariate diffusion process still follows the original Black and Scholes (1973) and Merton (1974) assumption of a lognormal diffusion with constant volatility.[7] This is all the more surprising, in view of the fact that the option pricing literature has long recognized that such an assumption is no longer adequate to represent underlying assets in option markets, and has introduced factors such as rare events, stochastic volatility and transaction costs. Choi and Richardson (2009) studied the conditional volatility of the firm's asset by a weighted average of equity, bond and loan prices and found that asset volatility is time varying. Hilberink and Rogers (2002) and Chen and Kou (2009) extend the Leland (1994b) model by incorporating a Levy process with only upward jumps and with two-sided double exponential jumps[8], respectively. In the study of the term

[3] For the reduced form models see Jarrow and Turnbull (1995), Duffie and Singleton (1999) and Duffie and Lando (2001). These models lie outside the topic of this paper.

[4] Under the first-passage default assumption, a firm will claim default when the asset value first crosses the pre-determined default boundary. This default boundary can be determined endogenously (Leland, 1994a,b, Leland and Toft, 1996) or exogenously (Longstaff and Schwartz, 1995).

[5] Zhou (2001) and Collin-Dufresne and Goldstein (2001).

[6] Brennan and Schwartz (1978), and more recently Titman and Tsyplakov (2007) are examples of studies that rely on numerical simulations.

[7] Most structural models are univariate and assume a constant riskless rate of interest. Longstaff and Schwartz (1995), Briys and de Varenne (1997), and Collin-Dufresne and Goldstein (2001) use bivariate diffusion models, in which the term structure of interest rates follows the Vacisek (1977) model and the asset value is a constant volatility diffusion. As the empirical work in Chan *et al* (1992) shows, the Vacisek model does not fit actual term structure data. Further, Leland and Toft (1996) note that this bivariate diffusion refinement plays a very small role in the yield spreads of corporate bonds.

[8] Zhou (2001) was the first to introduce jumps into structural models under the first passage default assumption, but no analytical solution is presented and he did not study the impact on optimal capital structure with endogenous

structure of credit default swaps (CDS), Huang and Zhou (2008) note that time varying asset volatility should potentially play a role in structural models in order to fit into the empirical credit default spread data. Huang (2005), and Zhang, Zhou and Zhu (2008) incorporate stochastic volatility and jumps into the Merton (1974) model by assuming that default occurs only at maturity and find that incorporating jumps and stochastic volatility may help to improve the matching of the top quality credit spreads.

In this paper we generalize the dynamics of the asset value by assuming that the diffusion volatility is state-dependent, varying with the asset value, and that the dynamics include an independent jump component with multinomially distributed amplitudes. We use a particular form of volatility state dependence, known as the Constant Elasticity of Variance (CEV) model, originally formulated by Cox (1975) in the context of option pricing.[9] Compared to constant volatility diffusion, the CEV model has an extra parameter and includes constant volatility as a special case. Although this extension introduces significant additional computational complexity, we manage to derive closed form expressions for almost all the variables of interest in the absence of jumps, including corporate debt value, total levered firm value, optimal leverage and equity value, while we obtain quasi-analytical numerical solutions for these same variables when jumps are present. Both Leland (L, 1994b) and Leland and Toft (LT, 1996) are special cases of the CEV model with zero elasticity of variance and no jumps. As a result of the flexibility provided by the extra parameters, our structural model under the CEV process with jumps is able to produce numerical results that are considerably closer to the historical record of yield spreads and default probabilities than the earlier structural models.

In the presence of jumps it is no longer possible to develop analytical expressions for the default probability distribution, but we do develop a numerical evaluation of the asset value distribution based on the inversion of the characteristic function. Based on this derived distribution, we also develop a numerical algorithm for the discrete time approximation of the distribution of the first passage time to default, which for computational accuracy purposes is restricted to occurring at 6-month intervals; for the simple CEV model we show that this restriction has a very slight effect on the main variables of interest. On the other hand, we show that the presence of jumps, even idiosyncratic risk ones, has a major impact on the credit spreads, the cumulative default probabilities and the structure of volatilities.

For the CEV model our numerical results depend strongly on the sign and value of the elasticity of variance parameter. A negative elasticity reduces the endogenous bankruptcy threshold for all debt maturities, and the downward shift increases with the absolute value of the elasticity. Similarly, a negative elasticity increases the optimal leverage for all maturities, with the increase being particularly pronounced for longer term debt structures. Consistent with its role as a risk-increasing feature of the model, a negative elasticity increases credit spreads for all but the largest leverage ratios, increases the effective duration of the bonds and increases sharply the volatility of the equity for all but the largest equity values. Adding binomial jumps to the diffusion and CEV models has the additional effect of reducing firm and equity values, increasing leverage and increasing credit spreads.

Empirically, by extracting the implied volatilities from the reported cumulative default probabilities for various bond rating categories in Moody's recent historical data according to the L and LT models, we

default boundary. Huang and Huang (2003) also incorporate double exponential jumps into a structural model, but they only focus on corporate debt valuation and credit spread.

[9] See also Emmanuel and MacBeth (1982), Cox and Rubinstein (1985), and Schroder (1989).

verify the fact that these volatilities are not constant but increase sharply for short-term debt, contrary to the models' assumptions. This downward term structures of implied volatilities for historical CDP could explain the underestimated shorter-term default frequencies by structural models found in Leland (2004). Further, under the CEV structural model, we demonstrate that a negative elasticity parameter shows similar increases in volatilities for short maturities and may contribute to a more accurate modeling of reality. Such added flexibility of modeling is enhanced by the presence of jump component, which raise cumulative default probabilities and implied volatilities for all rating categories.

Since our extensions have implications for several strands of literature that have dealt with different problems in corporate finance, we review the key issues examined by the class of models that we generalize. All these issues can be dealt with the same type of integrated models of the levered firm that we examine. The main such issue is the capital structure choice, which originates in the classic Modigliani and Miller (1963) analysis of the levered firm in the presence of taxes, according to which capital structure is chosen as a trade-off between the tax advantage of debt and the costs of possible bankruptcy. The pioneering work in this area that comes closest to our own approach is that of Leland (1994a,b, 1998), and LT.[10] Since several authors have raised doubts on whether the trade-off approach can really be invoked to justify observed leverage ratios, several studies focused on agency problems between stockholders and debtholders, or stockholders and managers.[11] These and other related studies show clearly the importance of the structural models in linking the default probabilities and yield spreads to the capital structure decision, a linkage that is missing from the reduced form models.

As already noted, we use the asset value of the unlevered firm as the basic underlying process for the valuation of the various components of the balance sheet of the levered firm, following Leland (1994a, b) and LT. In a variant of the basic model, presented in Goldstein, Ju and Leland (2001), the firm value is estimated from the dynamics of the earnings before interest and taxes (EBIT), split between the claimholders and the government.[12] A direct modeling of the dynamics, division and valuation of the firm's cash flows would in principle also be possible in the CEV model, but it will need to confront the troublesome issue of the valuation of non-traded assets, which is beyond the scope of this paper, and which in earlier studies is either avoided or carried out only under the most elementary assumptions.[13]

Our basic model generalizes directly the assumed unlevered firm value asset dynamics of L, by replacing constant with a state dependent volatility, and also by adding jump components to it as an extension; we also examine alternative debt structures as robustness checks. Otherwise, we follow the general assumptions initially formulated by Merton (1974), in which default is triggered when the asset value hits a lower default-triggering threshold. While in Merton's model default could only take place at maturity, Leland (1994a, b, 1998) and LT adopted debt assumptions that allowed default to take place before maturity. All these models can be included in our CEV formulation, while the mixed jump-diffusion models that we present allow default at discrete predetermined times. To our knowledge, this is the first paper to relax the constant volatility assumption of the earlier studies and still derive closed form

[10] See also, Sarkar and Zapatero (2003), Ju *et al* (2005) and Titman and Tsyplakov (2007),
[11] See Mella-Barral and Perraudin (1997), Leland (1998), Morellec (2004), and Ju *et al* (2005).
[12] See also Sarkar and Zapatero (2003).
[13] Sarkar and Zapatero (2003, p. 38, footnote 1) avoid the issue, while Morellec (2004) assumes that agents are risk neutral and the approach of Goldstein *et al* (2001) is only suitable to constant parameter diffusion processes for the cash flows.

solutions under continuous coupon payment and first passage default assumptions, and also the first model to incorporate jumps at times other than debt maturity.[14]

We close this literature review by noting a variant of the reduced form models, which is particularly popular in the financial mathematics literature. In that stream the primary asset dynamics, in the familiar forms of diffusion or jump diffusion, are applied not to the asset value but to the equity returns, as in the option pricing literature.[15] The advantage of this approach is that the equity returns, unlike firm value, are observable and available in high frequency data. Its disadvantage is, as with the reduced form models, that it does not allow the modeling of the firm's balance sheet and the linkage of the default process with the firm's capital structure, and for this reason will not be pursued in this paper. There exist statistical methods by which the parameters of the asset value dynamics can be estimated from the observed dynamics of the equity value for any given model.

The rest of the paper is organized as follows. In Section 2 we present the dynamics and corresponding distributions of the value of the firm's assets with and without jump component. In Section 3 we describe the stationary debt structure assumption and derive the analytical solutions for the value of corporate debt under various maturity assumptions under an exogenous bankruptcy trigger with CEV-diffusion process. In Section 4 we derive the closed-form solutions for equity value and total levered firm value under the CEV model and an endogenous bankruptcy trigger. In Section 5 the numerical values of endogenous default triggers, optimal capital structure, credit spread, debt capacity, equity volatility, duration and convexity of corporate debt and agency cost of debt under the CEV model are presented according to our base case calibration and compared with structural models under the fixed volatility assumptions. In Section 6 we compute the cumulative default probabilities and the term structure of implied volatilities by fitting optimally the CEV parameters to the reported cumulative default probabilities for various bond rating categories in Moody's recent historical data. In Section 7 we present an efficient algorithm to value the corporate debt under discrete default assumption in the presence of jumps and examine the effects of jump components in combination with simple and CEV diffusion on capital structure, debt value, equity value, credit spread, cumulative default probabilities and term structure of implied volatilities. Section 8 concludes. Several extensions and proofs of propositions and lemmas are in the appendix.

2. Economic Setup

2.1 Jump Diffusion process of unlevered asset

Following Leland (1994a,b), we consider a firm whose assets are financed by equity and finite maturity debt with a tax-deductible coupon. As in all previous related literature, the values of the components of the firm's balance sheet are estimated as contingent claims of the state variable *V*, the value of the unlevered firm's assets representing its economics activities, which follows a mixture of a continuous diffusion process V^D with state-dependent volatility $\sigma(V^D)$ together with an independent Poisson jump process (the physical or *P*- distribution):

[14] This is the case covered by Zhou (2001).
[15] See, in particular, Carr and Linetsky (2006) and Campi *et al* (2009).

$$\frac{dV}{V} = (\mu - q - \eta\mu_J)dt + \sigma(V^D)dW + JdN \tag{2.1}$$

where μ is the instantaneous expected rate of return of asset; q is the payout rate to the asset holders, including coupon payments to debt holders and dividends to equity holders; η is the jump arrival intensity and μ_J the mean of the logarithm of the amplitude distribution, $\ln(1+J)$; $\sigma(V^D)$ is a state dependent volatility; W is a standard Brownian motion; and N denotes the number of Poisson jumps. The constant risk free rate is denoted by r. Under the risk neutral measure (Q-distribution), Equation (2.1) becomes,

$$\frac{dV}{V} = (r - q - \eta^Q\mu_J^Q)dt + \sigma(V^D)dW^Q + J^Q dN^Q \tag{2.2}$$

This mixed process continues until the asset value hits or falls below a threshold value, denoted by K, for the first time. In such a case, a default event will be triggered and liquidation comes in immediately. Assuming the absolute priority is respected, the bond holders will then receive $(1-\alpha)K$, while the equity holders receive nothing. The remaining of asset value that equals to αK is considered a bankruptcy cost.

We denote the bond maturity by T, and the first passage time when the asset value reaches the threshold value by τ. The asset value dynamics then become,

$$\begin{cases} \dfrac{dV_t}{V_t} = (r - q - \eta^Q\mu_J^Q)dt + \sigma(V^D)dW^Q + J^Q dN^Q, \text{ if } 0 < t < \tau < T \\ V_t = \min\{V_\tau, K\} \text{ if } 0 < \tau \le t < T \end{cases} \tag{2.3}$$

For our state-dependent volatility, we set $\sigma(V_t^D) = \theta(V_t^D)^\beta$, in which case the diffusion process in (2.3) becomes a CEV (constant elasticity variance or volatility) diffusion process.

2.2 The CEV-diffusion model distribution for the unlevered asset

In this subsection we'll examine the firms under the assumption that there are no jumps $(\eta = 0)$, in which case $V_t^D = V_t$ and there are closed form expressions for most variables of interest; the jump process will be introduced in the following subsection. Without the jump component (2.3) becomes

$$\begin{cases} \dfrac{dV_t}{V_t} = (r - q)dt + \theta V_t^{\beta+1} dW^Q, \text{ if } 0 < t < \tau < T \\ V_t = K \text{ if } 0 < \tau \le t < T \end{cases} \tag{2.4}$$

The parameter β, the elasticity of the local volatility, is a key feature of the CEV model. For $\beta = 0$ the model becomes a geometric Brownian motion with constant volatility. For $\beta > 0$ ($\beta < 0$) (the state-dependent volatility is positively (negatively) correlated with the asset price.[16] In equity markets, the well-known "leverage effect" shows generally a negative relationship between volatility and equity price. There are also some suggestions that the economically appropriate range is $0 > \beta > -1$,[17] even though empirical evidence in the case of the implied risk neutral distribution of index options finds negative values significantly below this range. Jackwerth and Rubinstein (1999) find that the unrestricted CEV model when applied to the risk neutral distribution extracted from S&P 500 index options is able to generate as good out-of-sample option prices as the better known stochastic volatility model of Heston (1993). Hereafter we shall adopt $\beta \leq 0$ without any further restrictions as our base case, with the case $\beta > 0$ left as an exercise.

The CEV model yields a distribution of the asset value V_T conditional on the initial value V_0 that has the form of a non-central chi-square $\chi^2(z,u,v)$, denoting the probability that a chi-square-distributed variable with *u* degrees of freedom and non-centrality parameter *v* would be less than *z*. The shape of this distribution is given analytically most often in terms of its complementary form 1-$\chi^2(z,u,v)$, denoting in our case the probability $V_T \geq v_T$. For $\beta < 0$ this probability is given analytically by,[18]

$$\Pr ob(V_T \geq v_T) = 1 - \chi^2(c,b,a) = \chi^2(a, 2-b, c), \tag{2.5}$$

Where

$$\begin{aligned} &a = \upsilon v_T^{-2\beta},\ c = \upsilon(V_t e^{(r-q)T})^{-2\beta},\ b = -\beta^{-1}, \\ &\upsilon = -\frac{2(r-q)}{\theta^2 \beta [e^{-2(r-q)\beta T} - 1]} \end{aligned} \tag{2.6}$$

This distribution is the equivalent of the lognormal when the volatility is constant. It has been tabulated and is easily available numerically. Several additional results hold about the $\chi^2(z,u,v)$ distribution when the parameter *u* is an even integer that can simplify the computations. Nonetheless, the main result necessary for the extension to the mixed jump diffusion process by using the chi-square distribution's characteristic function holds even for non-integral degrees of freedom.[19]

2.3 The mixed jump-CEV diffusion model distribution for the unlevered asset

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[16] As Emmanuel and Macbeth (1982, p. 536) were the first to point out, for $\beta > 0$ the local volatility becomes unbounded for very large values of *V*, and there are technical issues concerning the mean of the process under both the physical and the risk neutral distribution. This problem is solved by assuming that the volatility is bounded and becomes constant for *V* exceeding an upper bound; see Davydov and Linetsky (2001, p. 963), A similar lower bound when β is < 0 prevents the formation of an absorbing state at 0.

[17] See Cox (1996), and also Jackwerth and Rubinstein (1999), who term this model the *restricted* CEV. The arguments in favor of the restricted CEV model are mostly applicable to index options and will not affect our formulation.

[18] See Schroder (1989, p. 213-214).

[19] See Johnson *et al* (1995, p. 433).

In this section we derive a quasi-analytical form of the conditional distribution of the unlevered asset value V_T given the initial value V_t under the mixed process(2.1), the equivalent of (2.5)-(2.6) for the CEV diffusion if the riskless rate *r* is replaced by the instantaneous drift μ.[20] In order to obtain quasi-analytical solutions we shall also restrict the class of distributions of the amplitudes of the jump processes that we'll consider, to discrete multinomial jumps, presenting results only for the binomial case without loss of generality. The quasi-analytical form is derived by the inversion of the characteristic function of the distribution, for which efficient numerical procedures exist.

Let $L_i,\ i=1,...,n$ denote the amplitude of the i[th] jump given n jumps in the interval *[0,T]*, and let Y_T denote the jump component in that period, with Y_T^n the conditional value of Y_T, $Y_T^n = \prod_{i=1}^{n} L_i$. From (2.2) and the independence of the diffusion and the jump components, we have $V_T = V_T^D Y_T$. The following auxiliary result, proven in the appendix, will be necessary for the estimation of the distribution of V_T under the mixed process.

Lemma 1: *The characteristic function of the distribution of* V_T *is given by*

$$E[e^{i\omega V_T}] = E\left[E\left[e^{i\omega Z_T} \mid Y_T\right]\right] = E\left[\phi_D(i\omega y_T)\right] \tag{2.7}$$

Where $y_T = Y_T^{-2\beta}.,\ Z_T = V_T^{-2\beta}$, *and*

$$\phi_D(i\omega) = E[e^{i\omega Z}] = \frac{\exp(\frac{i\omega c}{1-2i\omega})}{(1-2i\omega)^{\frac{b}{2}}} \tag{2.8}$$

and the parameters c, b are given by (2.6).

From this result we can now derive the distribution of V_T under the mixed process under a binomial distribution, by inverting the characteristic function given by Lemma 1. The following result is also proven in the appendix.

Proposition 1*: Let* $l_j = L_j^{-2\beta}\ \ j = u,d$ and *a given by* (2.6). *Then the probability distribution of* V_T *is given by*

$$\Pr ob(V_T \le v_T \left| V_t \right.) = \sum_{j=0}^{\infty} e^{-\eta T} \frac{(nT)^j}{j!} \sum_{i=0}^{j} \binom{j}{i} p_u^i (1-p_u)^{j-i} \Pr ob(Z_T \le a \mid N=j, y_T = l_u{}^i l_d{}^{j-1}) \tag{2.9}$$

Where

[20] Unlike diffusion, for which the transition from the *P*- to the *Q*-distribution is straightforward, the jump parameters of the mixed process are transformed in this transition, unless the jump risk is unsystematic. See also Section 7.

$$\Pr ob(Z_T \le a \mid N=j, y_T = l_u{}^i l_d{}^{j-1}) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}(e^{-i\omega Z_T}\phi_D(i\omega y_T))}{\omega}$$
$$= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}(e^{-i\omega Z_T}\phi_D(i\omega l_u{}^i l_d{}^{j-1}))}{\omega} \tag{2.10}$$

Although this distribution is not given in closed form, the expressions (2.9)-(2.10) yield quasi-analytical expressions for it under the binomial jump assumption, which easily and obviously generalizes to a multinomial one. In Section 7 we'll use these expressions in order to find the value of corporate debt under an exogenous default boundary under the Leland (1994b) model.

3. Corporate Debt Valuation Under the CEV Model

3.1 Stationary debt structure

We consider a claim such as a corporate bond on this underlying asset, denoted by $F(V,t)$). This claim pays a continuous nonnegative coupon C per unit time as long as the firm is solvent[21]. According to Black and Cox (1976), under a general state dependent volatility the following partial differential equation (PDE) has to be satisfied when the firm finances the net cost of the coupon by issuing additional equity, with the subscripts denoting partial derivatives and τ denoting the default time.

$$\begin{cases} \frac{1}{2}(\sigma(V))^2 V^2 F_{VV} + (r-q)VF_V + F_t + C - rF = 0 \ \text{ if } \ 0<t<\tau<T \\ F(V,t) = (1-\alpha)K \ \text{ if } \ 0<\tau\le t \end{cases} \tag{3.1}$$

A closed form solution for this equation for debt claims that are generally time-dependent is not available even under constant volatility. For this simpler case L and LT adopted particular debt maturity and repayment structures that allowed the solution of (3.1) as if the value of the debt claims were time-independent. In this paper we formulate and solve both the Leland (1994a,b) and the LT models, but we use L as our base case, since this model, with its exponential stationary debt structure, generates the most elegant results[22]. We assume that the debt has a total principal value P at time 0 when it is issued with coupon rate C. As time goes by, the firm retires this debt at a proportional rate g. Thus, the remaining principal value of this debt value at time t is $e^{-gt}P$, and the debt holders receive a cash flow $e^{-gt}(C+gP)$ at time t, provided the firm remains solvent. Hence, the average maturity of this debt will be, given that no default occurs,

$$T_a = \int_0^\infty gte^{-gt}dt = g^{-1} \tag{3.2}$$

[21] The firm is solvent only when the asset value is above the threshold value for bankruptcy all the time and never below it, starting from the issue date of this corporate bond.

[22] Compared to LT, the L model yields a simpler analytical solution. The debt service rate is $C+gP$ under L, while it is $C+P/T$ under LT. The two models are fully consistent with each other in their results if L's retirement rate g is changed to match the average maturity of debt structure under LT; see the LT case in the appendix.

Thus, the average maturity under the L model is the reciprocal of the proportional retirement rate. In order to get a stationary debt structure we assume that the firm replaces the retired debt with newly issued debt having the same principal and coupon so as to keep the total principal and total coupon payments independent of time. We denote the total value of all the outstanding debt by *D (V)*. Because all outstanding debts are homogenous, the initial total principal *P*, the coupon rate *C*, and the retirement rate g (or equivalently, the average maturity T_a) define the debt characteristics and can be used at time *0* as control parameters to value all the outstanding debt. When the volatility of the unlevered asset value is constant L derived this value *D(V)* analytically. If the volatility is state dependent then the solution of the corresponding PDE depends on the structure of state-dependent volatility. In the following section we use results from option pricing to derive an analytical solution for a particular case of state dependent volatility, when $\sigma(V) = \theta V^{\beta}$, which is the CEV form of the volatility.

3.2 The value of corporate debt with infinite maturity

We assume that the claim in PDE (3.1) is a risky corporate debt. In order to value this risky debt under a state-dependent volatility setting, we start from the simple case in which the maturity of this debt is infinite. Generally, for a risky perpetuity, the price is,

$$d(V) = [1 - p_d]\frac{C}{r} + p_d(1-\alpha)K, \text{ where } p_d = \int_0^{\infty} e^{-r\tau} 1_{\{\tau<\infty\}} d\tau \tag{3.3}$$

Where $1_{\{\tau<\infty\}}$ indicates one dollar payment when default occurs at time τ for the first time and C is the stationary coupon payment. Therefore, the first term in equation (3.3) gives us the present value of total expected coupon payments as long as the firm is solvent and the second term gives us the present value of the recovery payment when the firm is insolvent.

The value p_d is fundamental to our valuation approach. Since this value is also that of a down and out barrier option, we can use available results from option pricing to prove the following lemmas.

Lemma 2: *Under a general state dependent volatility* $\sigma(V)$ *the value of a contingent claim paying one dollar when the asset value V hits the barrier K under the risk neutral distribution is given by*

$$p_d = \int_0^{\infty} e^{-r\tau} 1_{\{\tau<\infty\}} d\tau = \frac{\phi_r(V)}{\phi_r(K)} \tag{3.4}$$

where $\phi_r(V)$ *is the decreasing fundamental solution of the following ordinary differential equation (ODE) for U(V,t),*

$$\frac{1}{2}\sigma(V)^2 V^2 U_{VV}(V,t) + (r-q)VU_V(V,t) - rU(V,t) = 0, V > 0 \tag{3.5}$$

Proof: See Proposition 1 of Davydov and Linetsky (2001).

When the volatility is constant, the above ODE has a solution $\phi_r(V) = V^{\gamma^*}$, where γ^* is the solution of a quadratic equation, yielding[23]:

$$\phi_r(V) = V^{-\gamma-\sqrt{\gamma^2+\frac{2r}{\sigma^2}}}, \gamma = \frac{r-q}{\sigma^2} - \frac{1}{2} \tag{3.6}$$

When the volatility is state-dependent, there is no analytical solution for the general form. We have, however, an analytical solution for the state-dependent variance under a CEV process given by

Lemma 3: *When the state dependent volatility is given by the CEV process $\sigma(V) = \theta V^{\beta}$, the solution of PDE* (3.5) *is given by*

$$\phi_r(V) = \begin{cases} V^{\beta+\frac{1}{2}} e^{\frac{\epsilon}{2}x} W_{k,m}(x), \beta < 0, r \neq 0 \\ V^{\beta+\frac{1}{2}} e^{\frac{\epsilon}{2}x} M_{k,m}(x), \beta > 0, r \neq 0 \end{cases} \tag{3.7}$$

Where,

$$x = \frac{|r-q|}{\theta^2|\beta|} V^{-2\beta}, \epsilon = sign((r-q)\beta), m = \frac{1}{4|\beta|}$$

$$k = \epsilon\left(\frac{1}{2} + \frac{1}{4\beta}\right) - \frac{r}{2|(r-q)\beta|}$$

$W_{k,m}(x)$ and $M_{k,m}(x)$ are the Whittaker functions.

Proof: See Proposition 5 of Davydov and Linetsky (2001).

The combination of equations (3.3), (3.4) and (3.7) may now be used to provide the analytical solution for the price of a risky corporate bond under our specifications when the asset value follows a CEV diffusion process. The Whittaker functions $W_{k,m}(x)$ and $M_{k,m}(x)$ are the fundamental solutions for the Whittaker equation and are available in the *Matlab* (or *Mathematica)* software.[24] Since the sign and value of β affect the probability of default by increasing (decreasing) the volatility in "bad" states when $\beta < 0$ ($\beta > 0$), the shape of $\phi_r(V)$ is also strongly affected by that parameter. It is a monotonic decreasing (increasing) function with respect to asset value V when $\beta < 0$ ($\beta > 0$). In addition, the slope of the function increases with the absolute value of β.[25]

3.3 The value of corporate debt under exponential debt structure and with finite maturity

[23] See Ingersoll (1987, p. 372).
[24] See Whittaker and Watson (1990, pp. 339-351).
[25] The relevant figures are available from the authors on request.

Under an exponential debt structure the weighted-average maturity of the risky corporate debt is T_a, where $g = 1/T_a$ from equation (3.2). At time *0* the firm issues perpetual debt with principal *P* and coupon payment *C*. We estimate the value of this debt with stationary structure as described in Sections 3.1. Since the debt payout rate is $e^{-gt}\left(C+gP\right)$ at time *t* and the debt holders' claim on the principal is $(1-\alpha)Ke^{-gt}$ in case of bankruptcy, the value of this debt at time 0 is,

$$D\left(V,K,g\right)=\int_0^\infty e^{-rt}\left[e^{-gt}\left(C+gP\right)\right]\left(1-A(t)\right)dt+(1-\alpha)K\int_0^\infty e^{-rt}e^{-gt}f(t,V,K)dt \qquad (3.8)$$

where $f(t,V,K)$ is the default time density function and $A\left(t\right)=\int_{\tau=0}^{\tau=t} f(\tau,V.K)d\tau$, the cumulative default probability from time 0 to *t*, omitting for simplicity its arguments. Integrating equation (3.8) by parts, we have,

$$D\left(V,K,g\right)=\frac{C+gP}{r+g}\left(1-\int_0^\infty e^{-(r+g)t}f\left(t,V,K\right)dt\right)+\left(1-\alpha\right)\int_0^\infty e^{-(r+g)t}f(t,V,K)dt \qquad (3.9)$$

where $\int_0^\infty e^{-(r+g)t}f(t,V,K)dt$ is the expected present value of one dollar payment when default takes place, with a discount rate equal now to $(r+g)$. Under a general sate dependent volatility process, we have the general form solution given by Lemma 1,

$$\int_0^\infty e^{-(r+g)t}f(t,V,K)dt=\frac{\phi_{r+g}(V)}{\phi_{r+g}(K)} \qquad (3.10)$$

where $\phi_{r+g}(V)$ is defined as the solution of equation (3.5) with a discount rate equal to $(r+g)$. Combining equation (3.9) and (3.10), we arrive at a general solution for the value of risky debt with maturity *t*,

$$D\left(V,K,g\right)=\frac{C+gP}{r+g}\left(1-\frac{\phi_{r+g}(V)}{\phi_{r+g}(K)}\right)+(1-\alpha)K\frac{\phi_{r+g}(V)}{\phi_{r+g}(K)} \qquad (3.11)$$

For the CEV process, we obtain a closed-form solution of (3.11) by replacing the function $\phi_{r+g}(.)$ by the expression (3.7) given by Lemma 3.

4. Equity Value and Endogenous Bankruptcy Trigger Under the CEV Model

4.1 Equity value and asset value

We derived an analytical solution for the value of the risky debt with stationary structure as in Section 3.3 under the CEV diffusion process by combining equation (3.11) and (3.7). In this section, we derive the value of the equity by valuing the tax shield due to the deductibility of the coupon interest and the bankruptcy cost. The corporate tax rate for the firm is denoted by w. As the interest paid to the bondholder is tax-deductible, the firm's total value is increased by the tax shield due to debt financing. However, the bankruptcy costs increase as well if the firm issues more debt to finance its projects. According to the trade-off theory, the manager of this firm should balance the tax benefit and the bankruptcy cost by maximizing the total firm value. This value can be expressed by,

$$v(V,K)=V+TB(V,K)-BC(V,K) \tag{3.12}$$

where $v(V,K)$ is the total firm value, *TB(V,K)* is the tax benefit due to debt financing and $BC(V,K)$ is the bankruptcy cost. For a risky debt with infinite maturity, the discount rate under the risk-neutral distribution to calculate the expected present value of one dollar when default occurs for the first time is the risk free rate. For a risky debt with finite maturity *T*, the discount rate will be the sum of the risk free rate plus the proportional retirement rate *g* that depends on the maturity of the debt. The tax benefit available to the firm equals the total tax benefit for a risk-free bond minus the tax benefit loss due to the default event[26], which yields,

$$TB(V,K)=\frac{wC}{r}-\frac{wC}{r}\frac{\phi_r(V)}{\phi_r(K)} \tag{3.13}$$

The bankruptcy cost is the present value of the loss due to the default event, equal to,

$$BC(V,K)=\alpha K\frac{\phi_r(V)}{\phi_r(K)} \tag{3.14}$$

Thus, we have,

$$v(V,K)=V+\frac{wC}{r}-\frac{wC}{r}\frac{\phi_r(V)}{\phi_r(K)}-\alpha K\frac{\phi_r(V)}{\phi_r(K)} \tag{3.15}$$

Since we assumed that the firm is financed by risky debts and equity, the value of the corresponding equity equals the total value of the firm minus the total value of the risky debts, which yields,

$$\begin{aligned} E(V,K)&=v(V,K)-D(V)\\ &=V+\frac{wC}{r}\left[1-\frac{\phi_r(V)}{\phi_r(K)}\right]-\frac{\phi_r(V)}{\phi_r(K)}\alpha K-\frac{C+gP}{r+g}\left(1-\frac{\phi_{r+g}(V)}{\phi_{r+g}(K)}\right)-(1-\alpha)K\frac{\phi_{r+g}(V)}{\phi_{r+g}(K)} \end{aligned} \tag{3.16}$$

4.2 The endogenous bankruptcy trigger

In the previous sections, we assumed that default happens when the state variable *V* drops below a default boundary, *K*. This default trigger value can be determined exogenously or endogenously. If a firm cannot

[26] We assume that the firm always benefits fully from the tax deductibility of coupon payments when it is solvent, as in Leland (1994a, b).

choose its default boundary value, then this boundary can be determined by a zero-net worth trigger[27] or by a zero cash flow trigger[28]. Under the zero-net worth trigger assumption, the default occurs when the net worth of the firm becomes negative for the first time, which implies that the default trigger value equals the total face value of the outstanding debt, namely $K = P$. However, we often observe that firms are still alive even though their net worth is negative in the financial markets. Thus, in order to improve the simple zero net worth trigger, Moody's KMV defines as trigger value $K = P_{Short} + 0.5 * P_{Long}$. Under zero cash-flow trigger, a firm claims default when the current net cash flow to the security holders cannot meet the current coupon payments, which implies $K = C/\delta$, where δ is the net cash flow to the security holders. The problem for this trigger value is that sometimes the equity value is still positive even though the current net cash flow is zero. In this case, a firm will prefer to issue more equity so as to meet the current coupon payment, instead of announcing default. On the other hand, if a firm is capable to choose its default boundary value, this default boundary value will be set endogenously by maximizing the total firm value. Following Leland (1994a) and LT, we may find the optimal endogenous default boundary by the smooth-pasting condition,

$$\frac{\partial E(V,K)}{\partial V}\Big|_{V=K} = 0 \tag{3.17}$$

This default boundary value maximizes the value of the equity at any asset level[29]. Applying (3.17) to (3.16), we get the following results, proven in the appendix.

Proposition 2*: According to the smooth pasting condition* (3.17)*, the endogenous default value under the CEV diffusion process, denoted by* K_e *, can be obtained by solving following equation for given parameter values* θ*,* β *and with the auxiliary variables defined in* (3.7)

$$1-\left[\frac{wC}{r}+\alpha K_e\right]\left[\frac{1}{\phi_r(K_e)}\frac{\partial \phi_r(K_e)}{\partial K_e}\right]+\left[\frac{C+gP}{r+g}-(1-\alpha)K_e\right]\frac{1}{\phi_{r+g}(K_e)}\frac{\partial \phi_{r+g}(K_e)}{\partial K_e}=0 \tag{3.18}$$

Where,

$$\frac{1}{\phi_r(K)}\frac{\partial \phi_r(K)}{\partial K}=\begin{cases}\dfrac{\beta+0.5}{K}+\left[0.5\varepsilon+0.5-\dfrac{k}{x}-\dfrac{W_{k+1,m}(x)}{W_{k,m}(x)x}\right]x', if\, \beta<0\\[2ex] \dfrac{\beta+0.5}{K}+\left[0.5\varepsilon+\dfrac{M_{k+1,m}(x)(k+m+0.5)}{M_{k,m}(x)x}-\dfrac{k}{x}+\dfrac{1}{2}\right]x', if\, \beta>0\end{cases}$$

Proposition 2 yields an endogenous default boundary by solving equation (3.18). Although there is no explicit solution for the endogenous default value, it is straightforward to find it from equation (3.18) by using a root finding algorithm, provided a positive root exists. We examine the properties of the solution, as well as the other variables of interest of the model in numerical examples in the following sections.

[27] See Brennan & Schwartz (1978), and Longstaff & Schwartz (1995).
[28] See Kim, Ramaswamy, &Sundaresan (1993).
[29] See Leland (1994a)

5. Numerical Analysis of the CEV Model

We analyze the impact of state-dependent volatility on endogenous default triggers, debt values, optimal capital structure and term structure of credit spreads by considering their values in a base case with the following parameters: current asset value $V = 100$, risk-free rate $r = 0.08$, firm's payout rate $q = 0.06$, tax rate $w = 0.35$, proportional bankruptcy cost $\alpha = 0.5$, and initial volatility of assets $\sigma_0 = 20\%$. Although some of these parameters may not reflect current conditions, they were chosen based on previous studies closely related to this paper, such as LT and Leland (2004), with which the results of this study need to be compared in order to assess the impact of the more general formulation. The remaining parameters will assume various values according to the studied topic.

5.1 Endogenous Default Triggers

[INSERT FIGURE 1 HERE]

Figure 1 shows the values of endogenous default boundaries for the L model, which is a special case of the CEV structural model when β equals zero, and four CEV structural models with $\beta = -1, -0.5, 0.5, 1$. In all cases the endogenous default trigger declines as the average maturity of the bond increases. This is consistent with the findings in the L and LT models. The economic interpretation is that the equity holder would rather sell equity to finance the required cash flow for debt servicing than choose to go bankrupt, even though the net worth of the firm may be negative, provided the anticipated equity appreciation is greater than the contribution required from the equity holders to keep a firm alive. For long term debt structure, the endogenous default boundary is usually less than the face value of the debt, which implies that the expected appreciation of equity for such a debt structure should be relatively higher than for the short-term debt structure. After incorporating the CEV process, we find that the smaller β is, the faster the endogenous default boundary declines. If β is negative and large in absolute value, and the average maturity of the debt is long enough, the corresponding endogenous default boundary could be close to zero, which implies that this firm would never choose to go bankrupt endogenously even though the net worth of the firm may be negative. On the other hand, if β is positive and large in absolute value, the endogenous default trigger of the CEV model decreases slowly and is greater than that of the L model.

These changes of endogenous default triggers under the CEV structural model can be understood economically from the point of view of the relationship between anticipated equity value and volatility. When β is negative (positive), the volatility of the asset increases (decrease) when the asset value decreases. Recall that it is well known since Merton (1974) that the equity in a levered firm can be interpreted as a call option on the value of the assets. Similarly, Merton (1973) showed that in many cases of underlying asset dynamics, including those used in this paper, the value of the option is an increasing function of the volatility. It follows that *ceteris paribus* the anticipated increase in the value of the equity will be inversely proportional to the value of β, while the default boundary will also vary inversely with this anticipated equity appreciation. In other words, at low values of *V* where the probability of default is high, the increase in volatility when β is negative will counteract the fall in the value of equity because of the fall in *V*.

[INSERT FIGURE 2 HERE]

Next we examine the effect of the leverage ratio, $D(V,K,g)/v(V,K,g)$, on the default boundary for two different values of maturity, or of its inverse *g*. The debt value at time 0 is set at par, implying that the RHS of equations (3.8), (3.9) and (3.11) is set equal to *P*. The L model shows a strictly increasing function of this endogenous default boundary with respect to the leverage ratio, depicted by the solid lines in Figure 2. As expected, the monotone increasing property of the default boundary as a function of the leverage ratio is preserved, but the speed of increase depends on the value of β. For the 20-year average debt maturity the shape of the function is convex for all β. The positive relationship between the value of β and the default boundary is maintained for negative β's at all leverage ratio, but not for positive β's, where we see a reversal for low values of the leverage ratio. Again, this is consistent with the option interpretation of the equity, since a low leverage ratio corresponds to a deep in the money call option, for which the volatility effect is weak and may be swamped by other factors in solving equation (3.18).

Note also that for the firms with a low leverage ratio and a negative β, there is a non-endogenous default zone in which a firm would never choose to go bankrupt endogenously, especially for the lowest value of $\beta = -1$. In Figure 2, this non-endogenous default zone with $\beta = -1$ starts from zero leverage and ends at 29% leverage for 5-year bonds and at 43% leverage for 20-year bonds. This shows a positive relationship between the range of the non-endogenous default zone and the average maturity of corporate bonds. By comparing the endogenous default triggers for CEV structural models with $\beta = -1$ and $\beta = -0.5$ we see the non-endogenous default zone becomes wider if β decreases. Again, this lowering of the default boundary to about zero is consistent with the volatility effect that causes *ceteris paribus* an appreciation of the equity treated as an option whenever the underlying value *V* decreases. In the non-endogenous default zone, the anticipated equity value is high enough to dominate the required cash outflow for debt required to keep the firm alive. Thus, the equity holders will choose to retire equity in order to fund the coupon payment for the debt holders until the equity value goes to zero. This behavior leads to lower recovery value for the bond holders, which increase the risk of corporate bonds.

5.2 Optimal capital structure

Under an endogenous default boundary and a pre-determined debt structure, the optimal capital structure that maximizes the total firm value can be achieved by altering the leverage ratio, the ratio of the total outstanding debt value over the total firm value, D/v. Figure 3 examines the relationship between total firm value and leverage ratio for bonds with average maturities from 1 to 20 years, and Table 1 reports the optimal leverage ratios and the values of key endogenous variables at optimal leverage. As the average maturity increases, the optimal leverage ratio increases and the optimal total firm value increases as well. For instance, for 1-year debt, the optimal leverage ratios are 32.88%, 28.44% and 27.20% for $\beta = -0.5$, $\beta = 0$ and $\beta = 0.5$ respectively, while for 10-year debt, the corresponding optimal leverage ratios are 66.39%, 59.61% and 51.63%. This relationship was first reported by L and is still preserved under CEV volatility.

[INSERT FIGURE 3 HERE]

[INSERT TABLE 1 HERE]

Based on the results in Figures 1 and 2, we anticipate that the optimal leverage ratios are affected by the value of β for given debt characteristics. We also expect that negative values of β are more leverage-friendly. These do indeed turn out to be the case. For intermediate or long term debt structures, the optimal leverage ratios increase (decrease) with the absolute value of β when β is negative (positive). For instance, the optimal leverage ratio increases from 51.43% to 67.06% when β decreases from 0 to -1, and decreases to 40.79% when β increases to 1 for the 5-year average maturity debt. For short-term debt structures, less than or equal to 1-year, the optimal leverage ratios decrease first and then start to increase when β increases from 0 to 1.

Similarly, the total firm value and the total debt value are both increasing functions while the total equity value is a decreasing function of the average maturity of the debt under each scenario considered in Table 1. This effect is consistent with the trade-off theory, which balances the tax benefits and bankruptcy costs of the firm in order to maximize the total firm value. Given that the anticipated bankruptcy costs are invariant to debt maturity, in long term debt the anticipated tax benefits that accumulate over time should dominate the anticipated bankruptcy costs, thus increasing both optimal leverage and firm value. Conversely, the anticipated bankruptcy costs should dominate the anticipated tax benefits for short-term debt.

Under the CEV model the strong effect of β on the optimal leverage also has predictable effects on total firm value. If β is negative, the optimal leverage ratio and total firm value will increase compared to the L model in which β is zero. The value effect is, however, small. For instance, for a 5-year average maturity the total firm value increases by about 5.5% when β changes from 0 to -1, a much smaller change than the respective changes in the debt and equity values. On the other hand, for positive values of β the effects of β on firm value do not have a consistent sign, with the effect dependent on debt maturity. Again, the effects on total value are small even though the shifts in the composition of capital structure are significant.

Table 1 also examines the risk characteristics of the optimally levered firm with the equity risk measured by equity volatility and the debt risk measured by debt volatility and credit spread. For a given debt maturity, both equity volatility and debt volatility monotonically increase when β decreases; the same is true for the credit spread for all but the largest maturity. For all the debt maturities considered in Table 1, the largest equity risk and debt risk measured by both risk metrics is always reached when β equals -1, the smallest β in our calibrations. On the other hand, for any given value of β, the volatility of debt increases monotonically with the average debt maturity. The volatility of equity and the credit spread, however, first increase and then decrease with maturity for β equal to -1,-0.5 and 1, while it increases with maturity for β equal to 0.5 and 0. Apparently, the impact of the size of β on optimal debt financing is major.

5.3 Credit spread and debt capacity

We calculate the term structure of credit spreads ($C/D-r$) of newly issued debt for alternative leverage ratios shown in Figure 4. For a given maturity debt, a high leverage ratio implies a high credit spread. The humped term structure, which first increase and then decreases, can be observed clearly for moderate-to-high leverage ratios for all values of β. These patterns are consistent with the findings of the L and LT models. Under the CEV structural model credit spreads are higher for negative than for positive β for all maturities and under all the leverage ratios considered in Figure 4. The humped shapes of term structure for moderate-to-high leverage ratios are still preserved with state-dependent volatility. This credit spread is inversely proportional to p_d, the present value of one dollar paid to debt holders when default occurs, which is given by Lemmas 1 and 2 for the risk neutral distribution. Since the default boundary is endogenously determined for given β, leverage ratio and debt maturity, these are also the variables that affect the credit spread.

[INSERT FIGURE 4 HERE]

Debt capacity is the maximum value of total debt under endogenous default boundary. L and LT found that debt capacity falls as the volatility of asset value increases under their constant volatility models. For the CEV model, debt capacity increases when β is negative and decreases when β is positive. Figure 5 depicts the debt value as a function of the leverage ratio for debt with 5-year average maturity. For the three values of β in the figure, the maximal debt value tends to be reached at approximately equal leverage ratios, which lie between 80% and 90%. Since a firm with a negative β has a higher optimal leverage, such a firm would typically experience a high debt value and a high total firm value compared to a firm with a positive β with exactly the same leverage ratio.

[INSERT FIGURE 5 HERE]

5.4 Duration and convexity of corporate debt

For the debt portfolio, the Macaulay duration, which measures the percentage change of bond value with respect to the change of the risk free interest rate, is one of the most popular and simple ways to measure interest rate risk for bonds with no default risk. For coupon-paying corporate bonds with default risk, L and LT studied the relationship between effective duration, which measures the real change of bond value with respect to the change in the risk free interest rate, and Macaulay duration. They found that the Macaulay duration is much longer than effective duration as the leverage ratio (or credit spread) increases, which implies that the traditional duration-matching methods for immunization should be adjusted when using corporate bonds. Following L, the Macaulay duration is given by $1/(g+C/D)$, while the effective duration is equal to $\left(\partial D/\partial r\right)*(1/D)$. In Figure 6, we fix the leverage ratio at 50% and show the relationship between Macaulay duration and effective duration for the CEV model under different values of β. For a constant volatility (solid line), the Macaulay duration is generally longer than the effective duration for all maturities, but under the CEV (dashed line for $\beta=-1$ and dotted line for $\beta=1$), the Macaulay duration is much closer to the real effective duration for any given effective duration. This implies that the traditional duration-matching method should be more effective under state-dependent than under constant volatilities.

[INSERT FIGURE 6 HERE]

For default-free debt, the debt value is a convex function of the interest rate, which is a critical property for the traditional duration matching method. However, L and LT found that this convexity does not necessarily hold any longer for corporate risky debt under the constant volatility assumption, a result confirmed in the scenarios considered in Figure 7 (solid lines). Under the CEV model, the convexity relation may appear again depending on the value of β. In Panel A of Figure 7, both the positive β ($\beta = 1$, dotted line) and the negative β ($\beta = -1$, dashed line) show a convex relationship for the debt with 5-year average maturity and 40% leverage ratio. When, however, the leverage ratio increases from 40% to 50% in Panel B, only the negative β preserves the convexity relationship. Thus, a dynamic duration-convexity hedge strategy for a bond portfolio should be implemented differently for different asset volatility assumptions, debt maturity and leverage ratio. For instance, for 20-year average maturity and 50% leverage ratio constant volatility yields a bond value that is a concave function of the interest rate, while for a CEV process with $\beta = -1$, the traditional duration-convexity hedging strategy still works because the convexity relationship still holds.

[INSERT FIGURE 7 HERE]

5.5 Equity volatility

Most structural models assume asset dynamics following a diffusion process for the unlevered firm value. Since this is a non-tradable asset and an unobservable variable, we need to estimate the drift and volatility of asset value from observable data of traded assets, such as the stock or bond price. For the model presented in this paper the parameters of the diffusion process may be estimated by a maximum likelihood method initially proposed by Duan (1994) that yields the asset value and volatility from observed equity value. By using three different models of asset dynamics, Ericsson and Reneby (2005) show that this method has superior properties compared to other estimators.

Once the parameters of the diffusion process have been estimated by the maximum likelihood method, it is straightforward to get the equity volatility and equity value. Since the analytical solution for the value of the equity has been derived under the CEV structural model, the volatility is given by applying Ito's Lemma to equation (3.16), equal to

$$\sigma_{Equity} = \frac{\partial E(V)}{\partial V} \theta V^{\beta+1} \tag{4.1}$$

[INSERT FIGURE 8 HERE]

In Figure 8, we assume that the firm is optimally levered under an endogenous default boundary and we examine the relationship between equity value and equity volatility for the L model and CEV structural models with varying β's. The L constant volatility model (solid line) indicates a negative correlation between equity value and equity volatility. As we use the whole US market's average data for our calibrations, this negative correlation is consistent with the findings for the market index data, which is popularly known as the "leverage effect". Under the CEV model, the correlation between equity value and equity volatility depends on the value of β. The smaller β is, the stronger the negative correlation between equity value and equity volatility. For instance, the dotted line ($\beta = -0.5$) is steeper than the solid line ($\beta = 0$) and the dashed line ($\beta = -1$) is even steeper compared to the case $\beta = -0.5$. On the other hand, when β is positive and large enough, it can also indicate a positive relationship between equity value and equity volatility, as in the case that $\beta = 1$ (plus-dashed line). While for the market index the leverage effect has been well documented, for an individual firm the correlation between equity value and equity volatility could be positive or negative, depending on the particular firm's characteristics. Thus, the CEV model allows more flexibility and generalization than the constant volatility model, which is only a special case of CEV structural models.

5.6 Agency effects: debt maturity and asset substitution

We noted in Table 1 that the total firm value increases when the maturity of the debt becomes longer under optimal capital structure. Rationally then all firms should use long-term debt to finance their projects in order to maximize total firm value. Why is short-term debt still traded in the bond market? Leland and Toft (1996) answer this question by studying the asset substitution effect for different debt maturities. The asset substitution originated from Jensen and Meckling (1976) and refers to the effect that equity holders will try to transfer value from debt to equity by increasing the riskiness of the firm's activities. By analyzing the relationship between $\partial E/\partial \sigma$ and $\partial D / \partial \sigma$ for different levels of asset value, LT find that

"the existence of potential agency costs implies that firms with higher asset risk will shorten their optimal debt maturity as well as decrease their optimal amount of debt."

[INSERT FIGURE 9 HERE]

We re-examine the asset substitution effect in the context of the CEV model in Figure 9. This figure shows the sensitivity of equity value and debt value to the total asset risk, $\sigma = \theta V^{\beta}$, respectively for the maturities of 1-year, 5-year, 10-year and perpetual. When the signs of $\partial E/\partial \sigma$ and $\partial D / \partial \sigma$ are the same, the interests of equity and debt holders are positively correlated, indicating a zero asset substitution effect, and *vice versa*. The constant volatility case ($\beta = 0$) is our benchmark for each scenario. As maturity increases from 1-year to perpetual, the asset substitution effect increases for all the scenarios being considered, which is consistent with LT's findings. For $\beta = -1$ the asset substitution effects are more severe, especially for intermediate- and long-term maturity, compared to those of the benchmark cases. On the other hand, for $\beta = 1$, most of the time the interests of equity holders are in line with those of the bond holders, provided the asset value is greater than the corresponding bankruptcy trigger for each maturity, indicating that increasing asset risk will decrease both equity value and debt value simultaneously. These observed stylized factors imply that firms would use short-term debt and suboptimal amounts when their asset value follows a CEV process with negative correlation between asset value and asset volatility, the most commonly assumed feature of asset dynamics.

6. Cumulative Default Probabilities and Implied Volatilities under a CEV Diffusion Process

6.1 Historical default data

In this section we evaluate the capacity of the CEV structure to approximate available bond risk structure historical data. Such data can be under the form of corporate bond prices or yields, as in the empirical studies of Anderson and Sundaresan (2000) and Eom, Helwege and Huang (2004), or default probabilities as in Leland (2004). The advantage of focusing on default probabilities rather than bond prices is that the cumulative default probabilities (CDP) are not affected by additional factors, such as illiquidity, state tax, etc. The physical asset dynamics of the underlying asset value are the main driver of the cumulative default probability. By contrast, the risk neutral process is the one that enters into corporate bond pricing. Although the two processes yield different results, the volatilities should be exactly the same under both risk-neutral and physical distributions for the diffusion and jump-diffusion models with diversifiable jump risk.[30]

[INSERT FIGURE 10 HERE]

The data for the cumulative default probability (CDP) of bonds of various risk classes and maturities is given in Moody's[31]. Figure 10 exhibits the CDPs for Aa, A, Baa and Ba rated corporate bonds for the periods 1983-2008 and 1983-2010. Since the sub-prime financial crisis starts in 2008, these two data sets show the CDPs pre and post crisis. We only consider these four middle ratings because the default probabilities are rather low for Aaa bonds, while bond ratings lower than Ba are too risky. As the figure shows, the CDP term structure after the sub-prime financial crisis shifts upward and becomes much steeper relative to that before the crisis.

6.2 Term structure of implied volatilities of historical CDPs under the L and LT models

In order to examine the consistency of the L and LT models with the above data we examine the volatilities implied by the CDPs under the assumption that the CDP data represents an "average" firm for each rating class with characteristics corresponding to the averages reported by Moody's. Under both L and LT it is assumed that a firm's value follows a diffusion process with constant volatility under the risk-neutral distribution, as in equation (2.2) with a constant σ and jump intensity $\eta^Q = 0$. Then the cumulative default probability till time T is,[32]

$$F(T;\sigma) = N\left(h_1(T)\right) + \left(\frac{V}{K}\right)^{-2a} N(h_2(T))$$
$$a = \frac{r-q-0.5\sigma^2}{\sigma^2}, b = ln\left(\frac{V}{K}\right), h_1(T) = \frac{-b-a\sigma^2 T}{\sigma\sqrt{T}}, h_2(t) = \frac{-b+a\sigma^2 T}{\sigma\sqrt{T}} \tag{5.1}$$

The corresponding CDP under the physical distribution is found by replacing $r-q$ by the drift $\mu-q$ of the physical distribution. Thus given an observed CDP for a particular maturity, risk free rate, payout rate,

[30] When the jump risk is systematic the market is incomplete, the physical and risk neutral jump process parameters are different, there are infinitely many possible transformations corresponding to the same physical distribution, and total volatility may be affected. For an analysis of this case see Oancea and Perrakis (2010).
[31] Moody's "Corporate Default and Recovery Rates 1920-2008", and "Corporate Default and Recovery Rates 1920-2010".
[32] See equation (4) in LT or equation (21) in L.

current asset value and exogenous default boundary, *K*, the volatility σ_{im}^{D} implied by the simple diffusion is given by,

$$\sigma_{im}^{D}(T) = F^{-1}(T; b, \mu, q) \tag{5.2}$$

[INSERT TABLE 2 HERE]

We show in Table 2 the information from Moody's used to find the implied volatility (IV) in (5.2). The risk premium, risk free rate, payout rate and average leverage for Aa, A, Baa and Ba rated corporate bonds are shown in Panel A, with the average leverage ratio found from Moody's special comment in 2006.[33] The average leverage ratio increases when the corporate debt rating decreases. We use a constant payout rate 6%, risk free rate 8% and tax rate 35% for all the corporate bonds. In addition, we assume that the current asset values are the same and equal to 100 and that the risk premium is 4%[34] implying a 12% rate of return of underlying asset value for all the bonds. Leland (2004) shows that an exogenous or endogenous default boundary fits the observed default probabilities equally well provided default costs and recovery rates are matched. In this study exogenous default boundaries which equal the value of debt for the "average" firm of each rating are used to calculate the IVs. The debt value is set according to the historical average leverage ratio for each rating.

[INSERT FIGURE 11 HERE]

We compute the simple diffusion IVs for our samples and plot them in Figure 11, while the average IVs for all the scenarios considered are shown in Panel B of Table 4. As expected, the IVs after the sub-prime financial crisis are relatively higher than those before the crisis for all the ratings because of the higher CDP after the crisis. The figure shows clearly that the term structure of IVs is not flat for all rating categories, which conflicts with the constant volatility assumption of both the L and LT diffusion models. Compared to the average IV for each rating category, the IVs are significantly higher for short-term corporate debt for all rating classes and occasionally higher for longer-term debt. As functions of maturity, the IVs are sharply decreasing initially and then become flat, with a slight increase when maturity is long enough. Under this asymmetric "U" shape term structure of IV the IVs for medium-term debt, around 10 to 15 years, are the lowest for each rating category. This indicates that the risk of short-term and long-term debt issued by a firm is higher than the risk of medium-term debt. Comparing the average of IVs in Table 2, we find that the sub-prime financial crisis does increase the volatility of asset value for all debt categories. As expected, the average IV increases with the average maturity of debt for all rating categories, since the risk of a firm is generally higher when it chooses to finance itself with longer term debt. On the other hand, the volatility does not always increase when the debt rating deteriorates, although the lowest rating has a much higher average volatility than the highest one; the slight drops in average volatility in intermediate ratings may be due to the composition of the sample. The U-shaped curve implies that if we use the average IV to predict the cumulative default probabilities of corporate debt for each rating, we will sharply under-estimate the default probability for short term and maybe slightly under-estimate the default probability for long-term. This is consistent with the Leland

[33] Moody's special comments: "The Distribution of Common Financial Ratios by Rating and Industry for North American Non-Financial Corporations: July 2006".
[34] A 4% asset risk premium is consistent with an asset beta of about 0.6, as used in Leland (2004).

(2004) findings.[35] As most curvature of the IV appears in the short term CDPs, we only focus on the short term CDPs in the following analysis, ranging from 1-year to 10-year CDPs.

6.3 Term structure of implied volatilities under the CEV structural model

Compared to the L and LT models, the CEV structural model has one extra parameter β to capture the state-dependent volatility of asset value. Can the CEV structural model generate the downward sloping term structure of IV? In order to answer this question, we first try to fit the historical term structure of IVs by varying β and θ and keeping the rest of the calibrations the same as in the LT model, which is used in most empirical work .[36]

[INSERT TABLE 3 HERE]

[INSERT FIGURE 12 HERE]

Table 3 reports the values of β and θ which minimize the sum of absolute deviations between IVs from the CEV structural models and IVs of historical CDPs for Aa, A, Baa and Ba rated bonds during the periods of 1983-2008 and 1983-2010. To keep the optimization problem simple, we only consider integer values of β. Compared to the LT model with flat term structure of IVs, the CEV structural model can generate a downward sloping term structure of IV, which can be visualized in Figure 12 for different rating categories during the periods we considered. At the same time, the sums of absolute differences between CEV IVs and Moody's historical IVs are small compared to those between LT IVs and Moody's historical IVs, especially for higher rated debt. For instance, the sum of the absolute errors of Aa bonds decreases by 85%, from 0.3146 to 0.0498 after incorporating the state-dependent volatility. Even for the Ba bond which has the highest β in our sample, the sum of the absolute errors still decreases by 25%, from 0.1208 to 0.0884. Therefore, the prediction of CDPs could be improved dramatically after introducing state-dependent volatilities to the asset value diffusion process, especially for higher rated bonds. Across different rating categories, β increases and the initial volatility, $\sigma_0 = \theta S^{\beta}$, increases when the bond rating decreases.[37] This indicates a higher volatility increase of higher rated bonds per unit decrease of asset value compared to that of lower rated bonds. Although the absolute scale of CDPs of higher rated debt is small, especially for short term maturities, the change of the IVs is relatively greater across different maturities. This evidence is consistent with Coval, Jurek and Stafford (2008)'s findings that although default risk is less important in an absolute sense for senior CDO tranches, systematic risk is extremely important as a proportion of total spreads for these tranches. In additional, by comparing the β and σ_0 with the period of 1983-2010, we found that the sub-prime financial crisis makes the term structure

[35] Note, however, that the evidence for the LT model from the sample of 182 firms used by Eom, Helwege and Huang (2004) to test the predictions of five structural models finds that LT *overpredicts* the yield spread for low maturity bonds.

[36] Since Leland (2004) shows that an exogenous or endogenous default boundary fits the observed default probabilities equally well provided default costs and recovery rates are matched, the exogenous default boundary is used here to keep the exercise simple.

37 As we assume the exogenous default boundary equals the face value of the debt so as to keep the calculation simple, it makes the initial volatilities relatively small. The value of the implied initial volatilities would increase and the value of β would not change when the exogenous default boundary decreases.

of IVs of Aa and A bonds much steeper and has relatively little impact on the term structure of IVs of Baa and Ba bonds.

7. The Mixed-Jump CEV Diffusion Process

7.1 Corporate debt valuation

As in the previous section, the valuation of the debt under the mixed process must be done under the Q-distribution, which will be assumed known, even though its extraction from observable market data is not a trivial exercise, We assume, for notational simplicity, that the jump risk is unsystematic, in which case the jump component's distribution remains unchanged and a contingent claim on the firm's assets must satisfy the following counterpart of (3.1)

$$\begin{cases} \frac{1}{2}(\sigma(V^D))^2 V^2 F_{VV} + (r-q-\eta\mu_J)VF_V + F_t + \eta E[F(V(1+J)-F] + C - rF = 0, \text{if } \ 0<t<\tau<T \\ F(V,t) = (1-\alpha)\min\{V_\tau, K\}, \text{if } \ 0<\tau\le t \end{cases} \qquad (6.1)$$

Since the integration of (6.1) is not feasible, we adopt again the same approach for debt valuation as for the simple CEV process. For this we need to develop the counterparts of equations (3.4) and (3.8)-(3.11) for the mixed process. These will be developed in a quasi-analytical format, since they are derived from the unlevered asset value distribution of Proposition 1, which is available in the same type of format.

The key to developing the equivalent of equation (3.11) for the jump diffusion model is the default time density function $f(t,V_t \mid 0,V_0)$, the first passage time probability distribution for this model for a given V_0 at time 0. A closed form expression for $f(t,V_t \mid 0,V_0)$ or for the value of a risky perpetuity similar to (3.4) does not exist for the mixed process. Instead, we develop an algorithm based on time discretization, as in the solution of Fortet's equation,[38] using the distribution of V_T derived in Proposition 1. Given $F(T,K \mid t,V_t) = \Pr ob(V_T \le K \mid V_t)$, we set $T = t+\Delta t$, and evaluate from (2.9)-(2.10) the probability $F(t+\Delta t, K \mid t, V_t)$ and the associated density $f(t+\Delta t, V_{t+\Delta t} \mid t, V_t)$ for values of V_t and $V_{t+\Delta t}$ in the two-dimensional array $[K,\infty)$. Define also, for $\Delta t = \frac{T}{n}$,

$$\begin{aligned} f_1(\Delta t, V_{\Delta t}) &\equiv f(\Delta t, V_{\Delta t} \mid 0, V_0) \\ f_k(k\Delta t, V_{k\Delta t}) &= \int_K^\infty f(k\Delta t, V_{k\Delta t} \mid (k-1)\Delta t, V^s_{(k-1)\Delta t}) f_{k-1}((k-1)\Delta t, V^s_{(k-1)\Delta t}) \, dV^s_{(k-1)\Delta t} \end{aligned} \qquad (6.2)$$

for $k \in [2,n]$. We then have the following result, whose proof is obvious and is omitted.

[38] See Colin-Dufresne and Goldstein (2001). Fortet's equation cannot be used here, either in its continuous or in its discrete time format, since the asset value path is discontinuous and at default the asset value may be less than K.

***Lemma 4**: If $\tau \in (0,T]$ denotes the first passage time to default, $G(T|0,V_0) = \Pr ob\{\tau \le T|0,V_0\}$ denotes its distribution, and Q_i denotes the probability that the firm asset value will lie below its default value K at $i\Delta t$ given that it lies above it at $j\Delta t,\ j = 1,2\cdots i-1$, these probabilities are given by the relations*

$$Q_1 = F(\Delta t, K|0,V_0), Q_2 = \int_K^\infty F(2\Delta t, K|\Delta t, V_{\Delta t}) f_1(\Delta t, V_{\Delta t}) dV_{\Delta t}$$
$$Q_i = \int_K^\infty F(i\Delta t, K|(i-1)\Delta t, V_{(i-1)\Delta t}) f_{i-1}((i-1)\Delta t, V_{(i-1)\Delta t}) dV_{(i-1)\Delta t},\ i \in [2,n] \tag{6.3}$$

and $G(T|0,V_0)$ can be approximated by

$$G(T|0,V_0) = \sum_1^n Q_i \tag{6.4}$$

Where $f_i(k\Delta t, V_{i\Delta t}),\ i = 1,...,n$ is given by (6.2).

A key issue in evaluating the results of interest in the structural CEV-jump model is the accuracy of the quasi-analytical solutions provided in Proposition 1 for sufficiently short time horizons necessary for the convergence of the numerical estimate of $G(T|0,V_0)$ to its continuous time limit, for which no analytical expression exists. This is clearly parameter-dependent and cannot be judged *a priori* in the absence of any standards for the continuous time solution when there are jumps. Further, the accuracy of the numerical estimates of $G(T|0,V_0)$ for the cases of simple and CEV diffusions, for which continuous time analytical solutions exist, can be achieved only for time intervals Δt as short as one week for reasonable parameter values, which are too short for accurate quasi-analytical solutions. For this reason we examine the effects of the presence of jump components under a simplifying assumption that default can occur *only* at half-year intervals, which coincide with coupon-payment dates for most cases of practical interest. Under such an assumption the quasi-analytical solution is accurate, and it is possible to compare the effects of jumps on the variables of interest for both simple and CEV diffusions under identical conditions.

Given now the first passage default probabilities under the mixed process and our simplifying assumption about default from (6.2)-(6.4), it is easy to estimate the debt values numerically from (3.8)-(3.9) for any given bankruptcy trigger *K*. In the following two subsections we present the numerical results for the CEV model with and without jumps under our simplifying default assumption.

7.2 Capital structure, debt value, equity value and credit spread

Since the first passage time distribution can be computed only numerically by means of the discrete time algorithm of Section 7.1, we also apply that algorithm to estimate the same distribution for the L and CEV models without jumps in order to focus on the jump effect, constraining the default occurrence to six-month intervals in all cases. Further, we set the *total* initial volatility (including the jump component) to be the same in all cases. Last, to minimize lengthy computations we use an exogenous default boundary, which we set equal to the optimal value of the unconstrained L and CEV diffusion cases in Table 1.

[INSERT TABLE 4 HERE]

We first compare the results of the L and CEV models with those of Table 1 under the same parameter choices and bankruptcy trigger in order to assess the effects of the restriction on the default times' occurrences. The debt and firm values were estimated from (3.11) and (3.12) respectively, while the equity value was found residually, by subtracting debt from firm value. As with the bankruptcy trigger, there was no attempt to estimate the optimal leverage ratio.

As we can see from the comparison of the entries in Table 1 with the corresponding no jump cases of Table 4, the default time restriction has a relatively small positive impact on the debt, equity and firm values, which becomes more important for the longer maturities. The impact is also small for the leverage ratio, which is very similar in all cases to the optimally chosen one in Table 1. The restriction does, however, have a significant effect on the credit spreads, which are uniformly higher in the unrestricted cases of Table 1, especially in the longer maturities; the difference in spreads reaches more than 35 basis points for the 10-year maturity and $\beta = -1$. This difference obviously reflects the difference in cumulative default probabilities when default is restricted at specific time points, which will be examined in the next section.

[INSERT TABLE 5 HERE]

Of considerably more interest is the examination of the effects of jumps on the simple and CEV diffusions, presented in Table 4 for the base case, while the effects of the jump distribution parameters are shown in Table 5 for the 5-year bond case. For the simple diffusion ($\beta = 0$) case, for which closed form expressions exist, we assume a lognormally distributed jump amplitude, while for the CEV cases we adopt the binomial model for this distribution. To maintain comparability we set the total initial volatility including the jump component equal under all scenarios. As we see from Table 4, the effect of the jumps is the opposite to that of the default time restriction, reducing firm value, equity and debt in all cases, especially for long-term debt, while increasing leverage slightly compared to that of the corresponding diffusion case. Note that these are not the optimal leverages after incorporating jump components since we use the optimal default boundary from Table 1 as the exogenous default boundary for each scenario, but these leverage ratios can still be considered as a measure of the total risk of a firm. In other word, the jump components will slightly increase the total risk of the firm compared to that of the corresponding diffusion case. As for the parameter effect, Table 5 shows that any parameter change that increases the riskiness of the firm such as an increase in the jump intensity or in the amplitude variance, or a decrease in the mean amplitude, results in a decrease in the firm, equity and debt values and an increase in the credit spread. The effects are more pronounced for the CEV model with $\beta = -1$ combined with binomial jumps, which show that a doubling of the jump intensity adds 11 basis points to the base case's credit spread, while raising the mean amplitude (ie. reducing the size of the downward jump) subtracts 15 points from that same spread. We conclude that the presence of jumps, even unsystematic ones, in the asset dynamics of the firm has a significant impact on its capital structure and credit quality, which varies with the frequency and severity of the downward jumps. The next section examines the impact of the jumps on the firm's default probabilities and term structure of implied volatilities.

7.3 Cumulative default probabilities and associated implied volatilities under the mixed diffusion-jump and CEV-jump processes

We note from Table 3 that the error minimizing value of β has to be very small in order to fit the empirically observed term structure of implied volatility for each bond rating under continuous time default. This is made feasible by the availability of an analytical solution under the pure CEV process derived in Section 3.3, which allows us to reach such small values of β in the empirical fitting. The lack of an analytical solution for the mixed-CEV jump precludes a numerical fitting of the parameters. Further, there are technical difficulties in inverting the characteristic function to get the cumulative default probabilities, as in Section 4. [39] For this reason we will only do a static analysis of the jump impacts on the cumulative default probabilities and associated implied volatilities, in which the lowest value of β is limited to -1.

Figure 13 and Figure 14 report respectively the term structure of CDPs and term structure of IVs for diffusion-jump and CEV-jump processes under varying calibrations. In order to make all the scenarios comparable, we set the initial asset value, $V_0 = \$100$, the exogenous default boundary at $K = \$50$ and the base case for jump at $\mu_J = -0.05, \sigma_J = 0.2, \eta_J = 1/10$. Both diffusion and CEV models with and without base case jumps are plotted by dashed and solid lines. For the CDPs, the presence of jump components shifts upwards sharply the default probabilities compared to that of the base case without jumps across all the models. The shift depends on the jump calibration. For instance, increasing jump intensity or the volatility of jump amplitude or decreasing the expected value of jump amplitude will increase the CDPs, even doubling the CDP for the longest maturity of 20 years compared to the no jump case when the intensity is equal to $\eta_J = 1/2$. As for the term structure of IVs shown in Figure 14, we find that the jump components shift the term structure of IVs upward from the no jump base case under all scenarios and also twist the shape of term structure according to the model. For the constant volatility case($\beta = 0$), the jump component would increase the IV for short term debt, not enough for our parameter choices to account for the observed downward term structure of empirical IV, but raising the possibility that a higher or a systematic jump risk may indeed explain the observed structure. For the CEV-jump models the jump component has a more noticeable impact on the long term maturity, especially for positive β .

We conclude that jump components, for all the difficulties that they present in deriving analytical solutions, have significant impacts on default probabilities and the associated volatility and credit spread metrics. The evidence that we present from our numerical algorithm can only be considered preliminary, and more research is needed, especially with respect to improving the accuracy of the derived solutions. Such improvements may allow a shorter discretization interval and, thus, bring the results of the algorithm closer to the unknown continuous time solutions.

8. Conclusion

[39] When the cumulative probability is very small, such as 1E-20,because of very short time periods or a very large asset value compared to the bankruptcy trigger, the errors in evaluating the integral from 0 to $+\infty$ are relatively too large to generate reasonable results.

We have presented a new structural model of the firm that generalizes the asset dynamics assumptions of Leland (1994a,b) and Leland and Toft (1996), among others. The generalizations are twofold. First, we introduce a state dependent volatility that varies with the underlying asset, the value of the unlevered firm, under the constant elasticity of variance form. We derive closed form expressions for almost all the variables of interest on the balance sheet, including corporate debt values, total levered firm values and equity values. Under the derived endogenous default triggers we study the impact of state dependent volatilities on default probabilities, optimal leverage, credit spreads, debt capacity, duration and convexity of corporate debt, and the agency effects of debt. Most of the results are given implicitly, but efficient numerical methods allow us to reach the solutions easily.

Second, we introduce jump components into both the simple and the state dependent diffusion dynamics. We derive quasi-analytically the asset value distribution under multinomial jumps and derive a discrete time algorithm for the first passage time distribution by restricting possible default time to every semester. Although the lack of analytical expressions prevents several important derivations, we nonetheless establish that the presence of even unsystematic jump risk increases significantly default probabilities and the term structure of default volatilities.

By comparing the simulated predictions of the extended models with those of the closely related Leland (1994b) and Leland and Toft (1996) models under the same parameter values apart from the state dependent volatility and the jump parameters, we find that the new parameters are major determinants of all the variables of interest. We also find that the extra parameters of the new models allow a significant degree of flexibility in order to match observed real life measures such as the cumulative default probabilities of bonds of varying rating classes.

An open question is whether the advantages of these additional parameters are maintained in out-of-sample predictions. This is an issue to be resolved in future empirical research.

Appendix A: The value of corporate debt under the CEV model and the LT debt assumptions

In LT's stationary debt structure the firm continuously sells a constant amount of new debt with maturity T and redeems the same amount of matured debt in order to keep the total outstanding principal and coupon payment rate constant and equal to P and C, respectively. Suppose $d(V,K,T)$ denotes the price of one unit of outstanding debt with finite maturity T, continuous coupon payment C, and principal P, which can be expressed by

$$d\left(V,K,t\right)=\begin{cases}\int_0^T e^{-rt}Cdt+e^{-rT}P, 0<T<\tau \\ \int_0^\tau e^{-rt}Cdt+e^{-r\tau}(1-\alpha)K, 0<\tau<T\end{cases} \tag{A.1}$$

Given the first passage probability density function $f(\tau,V,K)$ under the CEV process, we define, omitting arguments for notational simplicity

$$B\left(T\right)=\int_{\tau=0}^{T} e^{-r\tau} f(\tau,V,K)d\tau, \tag{A.2}$$

the expected price of one dollar payment when default occurs during the period (0,T). From equation (3) in LT we find the price of this bond equals

$$d\left(T,V,K\right)=\frac{C}{r}+e^{-rT}\left[P-\frac{C}{r}\right]\left[1-A(T)\right]+\left[\left(1-\alpha\right)K-\frac{C}{r}\right]B(T). \tag{A.3}$$

$A(T)$, the cumulative first passage default probability, was defined in Section 3.3. Under the CEV process analytical forms for $A(T)$ and $B(T)$ are given in the following Lemma.

***Lemma A1**: When the state dependent volatility is given by the CEV process $\sigma\left(V\right)=\theta V^{\beta}$, the first passage cumulative default probability, A(T) and the expected price of one dollar payment when first passage default occurs during the period from time 0 to T equal*

$$\begin{aligned}A\left(T\right)&=\mathcal{L}^{-1}\left[\frac{1}{\lambda}\frac{\phi_{\lambda}(V)}{\phi_{\lambda}(K)}\right]\\ B\left(T\right)&=\mathcal{L}^{-1}\left[\frac{1}{\lambda}\frac{\phi_{r+\lambda}(V)}{\phi_{r+\lambda}(K)}\right]\end{aligned} \tag{A.4}$$

Where $\mathcal{L}^{-1}$ denote the inverse of the Laplace transform evaluated at the appropriate debt maturity T and $\phi_{\lambda}(V)$ is defined in equation (3.7)

Proof: It suffices to prove the second part of (A.4) since the first part follows immediately by setting *r=0*. By definition the Laplace transform $\Lambda(\lambda)$ of *B(T)* is given by

$$\Lambda(\lambda)=\int_0^\infty e^{-\lambda T}\left[\int_0^T e^{-r\tau} f(\tau,V,K)d\tau\right]dT=\int_0^\infty e^{-\lambda T}E_\tau\left[1_{\tau\le T}e^{-r\tau}\right]dT \tag{A.5}$$

By changing the order of integration (A.5) becomes

$$\Lambda(\lambda)=\int_0^\infty\left[\int_\tau^\infty e^{-\lambda T}dT\right]e^{-r\tau} f(\tau,V,K)d\tau=\int_0^\infty\frac{1}{\lambda}e^{-(r+\lambda)\tau} f(\tau,V,K)d\tau \tag{A.6}$$

Since the RHS of (A.6) is a constant times the value of a $1 perpetual claim in the first passage time under the CEV distribution, (A.4) follows immediately from (A.6) by Lemmas 1 and 2, QED.

By inserting equation (A.4) into (A.3), we arrive at the solution for the price of risky corporate debt with finite maturity *T* under the CEV diffusion process. Thus, under the LT model's stationary debt structure, the value of all outstanding debt with maturity *T* is from (A.3),

$$\begin{aligned}D(V,K,T)&=\int_{t=0}^{T}d(V,K,t)dt\\&=\frac{C}{r}+\left(P-\frac{C}{r}\right)\left(\frac{1-e^{-rT}}{rT}-I(T)\right)+\left((1-\alpha)K-\frac{C}{r}\right)J(T)\end{aligned} \tag{A.7}$$

Where

$$I(T)=\frac{1}{T}\int_0^T e^{-rt}A(t)dt,\ J(T)=\frac{1}{T}\int_0^T B(t)dt$$

An analytical expression for $I(T)$ is

$$I(T)=\frac{1}{T}\int_0^T e^{-rt},A(t)dt=\frac{1}{rT}\left(B(T)-e^{-rT}A(T)\right)$$

Unfortunately there is no analytical solution for $J(T)$, which must be evaluated numerically from the function *B(T)* estimated by(A.4).

Note that for debt with maturity *T* the first passage probability density function, $f(\tau,V,K)$ should be exactly the same for both L and LT stationary debt structures under the CEV process. Hence, we may define the equivalent retirement rate g' as the one that makes the expected price of a one dollar payment when default occurs equal under both L and LT debt structures, or $B_L(g')=B_{L\&T}(T)$, for *T* corresponding to T_a and *g=g'* in (3.2).

$$\int_{\tau=0}^{T} e^{-r\tau} f(\tau,V,K)d\tau = \int_{0}^{\infty} e^{-\left(r+g^{'}\right)\tau} f(\tau,V,K)d\tau \tag{A.8}$$

Since the RHS of (A.8) is available analytically from the inversion of the Laplace transform in the second part of (A.4), the value of corporate debt under the LT stationary debt structure can be found for any T from (A.8) by applying the equivalent retirement rate $g^{'}$ and setting $T=g^{'-1}$.

Appendix B: Proof of Lemmas and Propositions

Proof of Lemma 1

The characteristic function (2.8) of a non-central χ^2 variable is a well-known result.[40] (2.7) follows then immediately from (2.6) and the definitions of Z_T and y_T, QED.

Proof of Proposition 1

The characteristic function inversion (2.10) is well-known; see, for instance, Heston (1993, p. 331). (2.9) follows then immediately by noting that $\Pr ob(V_T \le K) = \Pr ob(Z_T \le a)$ and that $\Pr ob(V_T \le K) = E[\Pr ob(V_T \le K \,|\, N = j, y_T = l_u^{\,i} l_d^{\,j-1})]$, QED.

Proof of Proposition 2

Under CEV diffusion process, the equity value can be found from equations (4.5),

$$\begin{aligned} E\left(V,K\right) &= v\left(V,K\right) - D\left(V\right) \\ &= V + \frac{wC}{r}\left[1 - \frac{\phi_r\left(V\right)}{\phi_r\left(K\right)}\right] - \frac{\phi_r\left(V\right)}{\phi_r\left(K\right)}\alpha K - \frac{C+gP}{r+g}\left(1 - \frac{\phi_{r+g}\left(V\right)}{\phi_{r+g}\left(K\right)}\right) - (1-\alpha)K\frac{\phi_{r+g}(V)}{\phi_{r+g}(K)} \end{aligned} \tag{B.1}$$

Where,

$$\phi_r\left(V\right) = \begin{cases} V^{\beta+\frac{1}{2}} e^{\frac{\epsilon}{2}x} W_{k,m}\left(x\right), \beta < 0, r \ne 0 \\ V^{\beta+\frac{1}{2}} e^{\frac{\epsilon}{2}x} M_{k,m}\left(x\right), \beta > 0, r \ne 0 \end{cases} \tag{B.2}$$

[40] See, for instance, Walck (2007, p. 110).

Where,

$$x=\frac{|r-q|}{\theta^2|\beta|}V^{-2\beta},\varepsilon=sign((r-q)\beta),m=\frac{1}{4|\beta|}$$

$$k=\epsilon\left(\frac{1}{2}+\frac{1}{4\beta}\right)-\frac{r}{2|(r-q)\beta|}$$

$W_{k,m}(x)$ and $M_{k,m}(x)$ are the Whittaker function.

Following Leland (1994a), the smooth pasting condition implies,

$$\frac{\partial E(V,K)}{\partial V}|_{V=K}=0 \tag{B.3}$$

We define,

$$\frac{\partial W_{k,m}(x)}{\partial x}=W^{'},\frac{\partial M_{k,m}(x)}{\partial x}=M^{'},\frac{\partial x}{\partial V}=x^{'}$$

From the Mupad notebook in Matlab software[41], we have,

$$\begin{aligned}W^{'}&=-\left(\frac{k}{x}-\frac{1}{2}\right)W_{k,m}(x)-\frac{W_{k+1,m}(x)}{x}\\ M^{'}&=\frac{M_{k+1,m}(x)(k+m+0.5)}{x}-(\frac{k}{x}-\frac{1}{2})M_{k,m}(x)\end{aligned} \tag{B.4}$$

From (B.1) we have,

$$\frac{\partial E(V,K)}{\partial V}=1-\left[\frac{wC}{r}+\alpha K\right]\left[\frac{1}{\phi_r(K)}\frac{\partial\phi_r(V)}{\partial V}\right]+\left[\frac{C+gP}{r+g}-(1-\alpha)K\right]\frac{1}{\phi_{r+g}(K)}\frac{\partial\phi_{r+g}(V)}{\partial V} \tag{B.5}$$

Where,

$$\frac{\partial\phi_r(V)}{\partial V}=\begin{cases}\left[\beta+0.5+0.5Vx^{'}\varepsilon\right]V^{\beta-0.5}e^{0.5\varepsilon x}W_{k,m}(x)+V^{\beta+0.5}e^{0.5\varepsilon x}W^{'}x^{'},if\,\beta<0\\ \left[\beta+0.5+0.5Vx^{'}\varepsilon\right]V^{\beta-0.5}e^{0.5\varepsilon x}M_{k,m}(x)+V^{\beta+0.5}e^{0.5\varepsilon x}M^{'}x^{'},if\,\beta>0\end{cases}$$

Applying the smooth pasting condition that sets the RHS of (B.5) to 0, we have

[41] The first derivative of the Whittaker function with respect to x can be found by command: diff(whittakerM(a,b,z),z) and diff(whittakerW(a,b,z),z)

$$1-\left[\frac{wC}{r}+\alpha K\right]\left[\frac{1}{\phi_r\left(K\right)}\frac{\partial\phi_r\left(K\right)}{\partial K}\right]+\left[\frac{C+gP}{r+g}-\left(1-\alpha\right)K\right]\frac{1}{\phi_{r+g}\left(K\right)}\frac{\partial\phi_{r+g}\left(K\right)}{\partial K}=0 \qquad \text{(B.6)}$$

Where,

$$\frac{1}{\phi_r\left(K\right)}\frac{\partial\phi_r\left(K\right)}{\partial K}=\begin{cases}\dfrac{\beta+0.5}{K}+\left[0.5\varepsilon+0.5-\dfrac{k}{x}-\dfrac{W_{k+1,m}\left(x\right)}{W_{k,m}\left(x\right)x}\right]x', if\,\beta<0\\ \dfrac{\beta+0.5}{K}+\left[0.5\varepsilon+\dfrac{M_{k+1,m}\left(x\right)(k+m+0.5)}{M_{k,m}\left(x\right)x}-\dfrac{k}{x}+\dfrac{1}{2}\right]x', if\,\beta>0\end{cases}$$

which corresponds to equation (3.18), QED.

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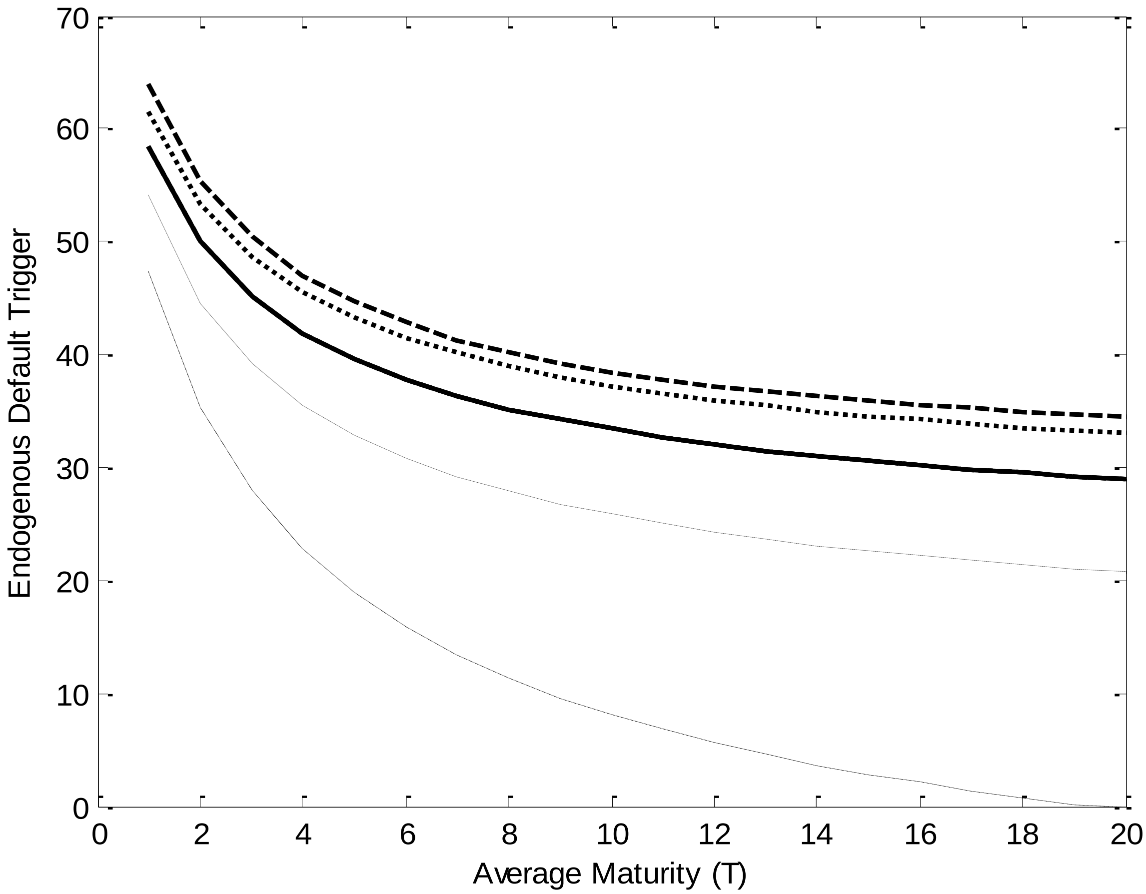

Figure 1: Endogenous Default Trigger as a function of average maturity. This figure depicts the values of endogenous default triggers for the Leland (1994b) model (bold solid line) and CEV structural models with $\beta = -1$ (dashed line), $\beta = -0.5$ (dotted line), $\beta = 0.5$ (Bold dotted line) and $\beta = 1$ (bold dashed line). It is assumed that current asset value $V = 100$, current debt value $D = 50$, risk free rate $r = 0.08$, the firm's payout rate $q = 0.06$, tax rate $w = 0.35$, and proportional bankruptcy cost $\alpha = 0.5$. The coupon rate is determined by making the debt issued at par value under the endogenous default boundary. The initial volatilities are the same for all these scenarios, and $\sigma_0 = 20\%$. For each given β under the CEV diffusion process, $\theta = \sigma_0 \ V^{\beta}$.

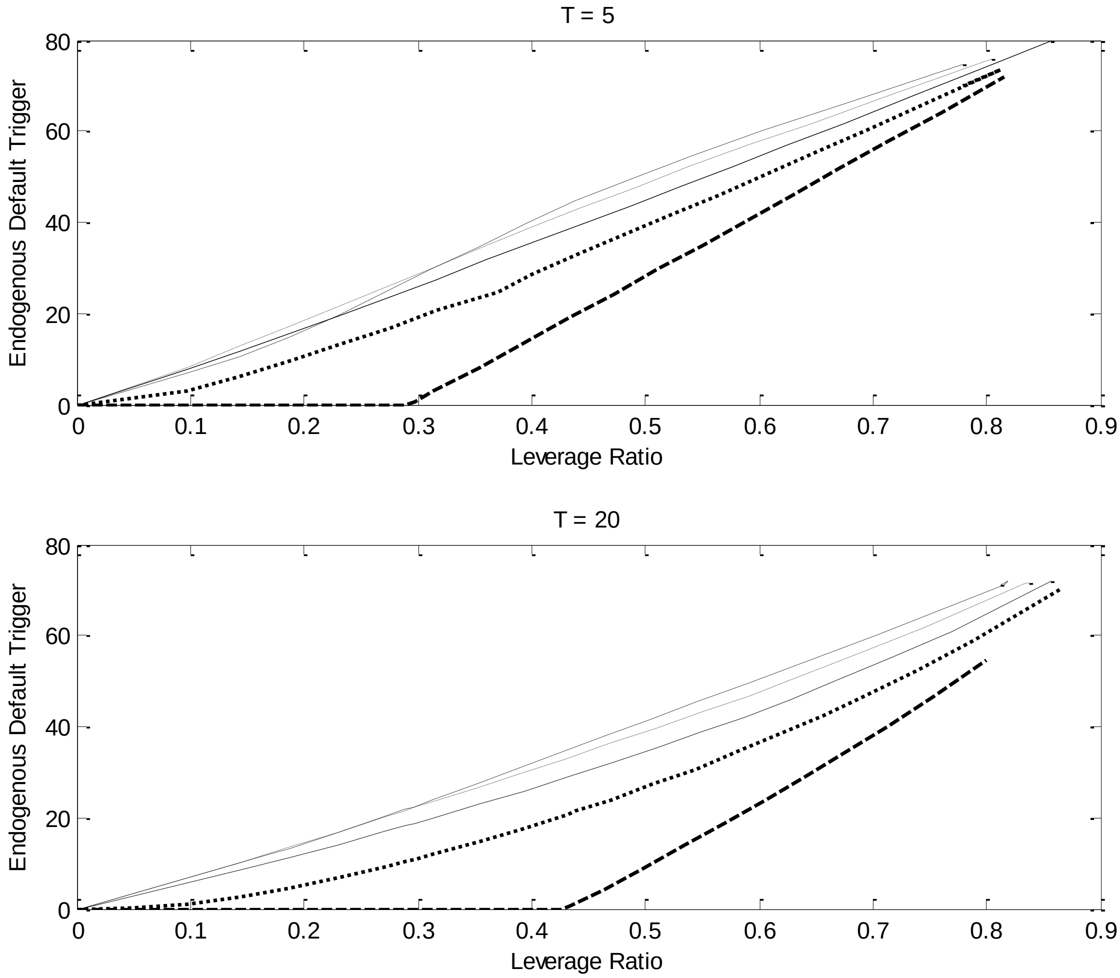

Figure 2: Endogenous default trigger as a function of leverage ratio. This figure depicts the value of the endogenous default trigger with respect to the leverage ratio under the Leland (1994b) model (solid line) and CEV structural models with $\beta = -1$ (Bolded dashed lines), $\beta = -0.5$ (Bolded dotted lines), $\beta = 0.5$ (dotted lines) and $\beta = 1$(dash lines). It is assumed that current asset value $V = 100$, risk free rate $r = 0.08$, the firm's payout rate $q = 0.06$, tax rate $w = 0.35$, and proportional bankruptcy cost $\alpha = 0.5$. The coupon rate is determined by making the debt issued at par value under the endogenous default boundary. The initial volatilities are the same for these scenarios and $\sigma_0 = 20\%$. For each given β under the CEV diffusion process, $\theta = \sigma_0 \ V^{\beta}$.

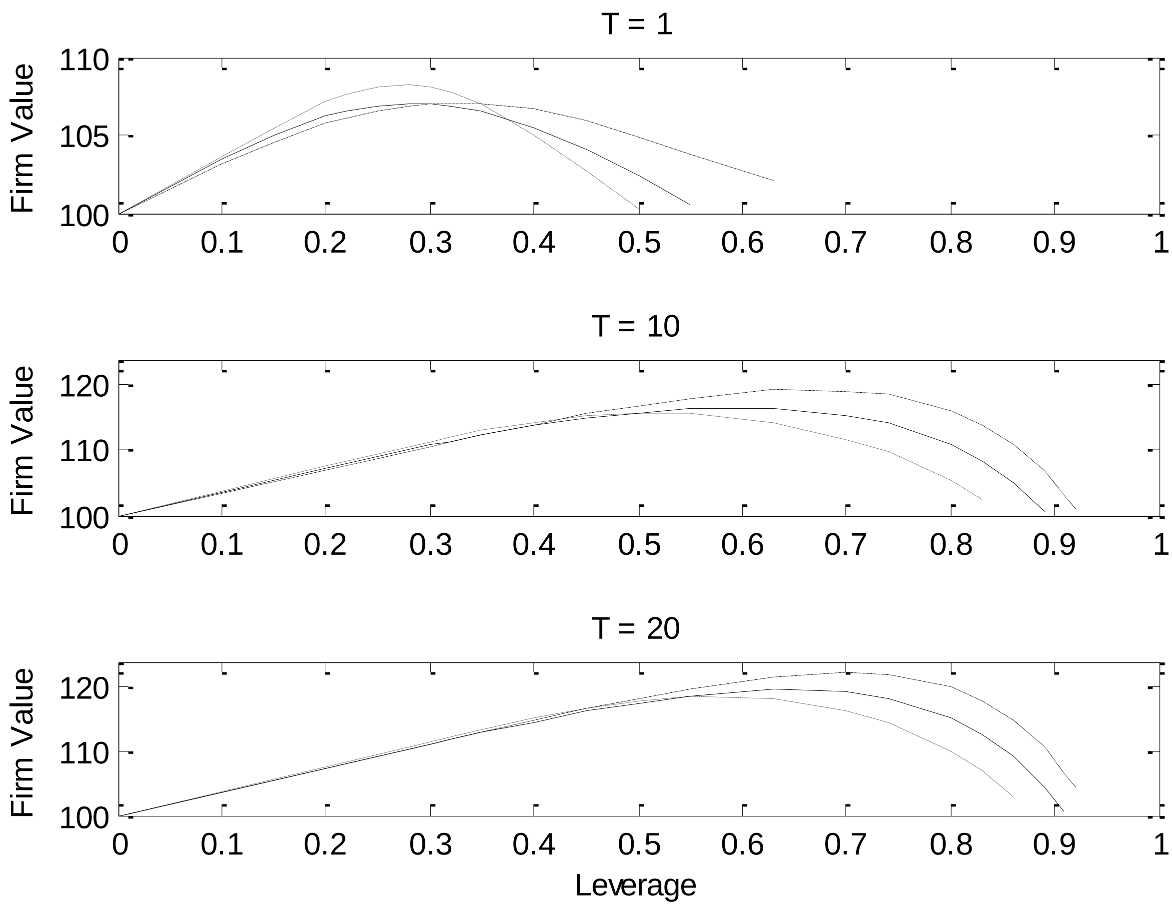

Figure 3: Total firm value as a function of leverage ratio. This figure depicts the total firm value with respect to the leverage ratio under the Leland (1994b) model (solid line) and CEV structure models with $\beta = -0.5$(dashed lines) and $\beta = 0.5$ (dotted lines). Three scenarios of average debt maturity are considered, $T = 1$, $T = 10$ and $T = 20$. It is assumed that current asset value $V = 100$, risk free rate $r = 0.08$, the firm's payout rate $q = 0.06$, tax rate $w = 0.35$, and proportional bankruptcy cost $\alpha = 0.5$. The coupon rate is determined by making the debt issued at par value under the endogenous default boundary. The initial volatilities are the same for these scenarios and $\sigma_0 = 20\%$. For the given β under the CEV diffusion process, $\theta = \sigma_0 \ V^{\beta}$.

Table 1: Characteristics of optimally levered firms under different models.

This Table exhibits the characteristics of optimally levered firms under the Leland (1994b), Leland and Toft (1996) , and CEV structural models with 1-year's, 5-years', and 10-years' average maturity. It is assumed that current asset value $V = 100$ dollars, risk free rate $r = 0.08$, the firm's payout rate $q = 0.06$, tax rate $w = 0.35$, and proportional bankruptcy cost $\alpha = 0.5$. The leverage is chosen by maximizing total firm value and the coupon rate is determined by making the debt issued at par value under the endogenous default boundary. The initial volatilities are the same for these scenarios and $\sigma_0 = 20\%$. For each given β under the CEV diffusion process, $\theta = \sigma_0 \ V^{\beta}$.

Models	Coupon (Dollars)	Bankruptcy Trigger (Dollars)	Optimal Leverage (Percent)	Firm Value (Dollars)	Equity Value (Dollars)	Total Debt Value (Dollars)	Equity Volatility (Percent)	Total Debt Volatility (Percent)	Credit Spread (Basis Points)
				1-year Average Maturity					
$\beta = -1$	3.59	37.52	39.05	108.59	66.18	42.41	35.63	0.71	48.31
$\beta = -0.5$	2.86	36.72	32.88	107.06	71.87	35.20	31.33	0.2	14.46
$\beta = 0.5$	2.35	36.25	27.20	108.19	78.76	29.42	26.26	3.04E-3	0.23
$\beta = 1$	2.54	38.76	28.90	110.00	78.21	31.80	25.94	3.24E-4	0.03
L($\beta = 0$)	2.44	35.67	28.44	107.06	76.61	30.45	27.99	3.12E-2	2.3
				5-year Average Maturity					
$\beta = -1$	8.70	51.83	67.06	119.44	39.34	80.10	58.40	8.90	286.30
$\beta = -0.5$	7.11	50.98	61.00	115.44	45.02	70.42	51.37	6.09	210.31
$\beta = 0.5$	3.99	41.72	42.75	112.74	64.54	48.20	32.85	0.67	27.51
$\beta = 1$	3.76	41.29	40.79	114.33	67.69	46.64	30.24	0.14	6.45
L($\beta = 0$)	5.23	46.36	51.43	112.99	54.88	58.12	40.82	2.69	100.51
				10-years Average Maturity					
$\beta = -1$	9.09	46.95	70.43	123.37	36.49	86.89	55.50	9.93	245.62
$\beta = -0.5$	8.07	49.51	66.39	119.38	40.12	79.26	52.70	8.06	218.09
$\beta = 0.5$	5.17	44.62	51.63	115.71	55.97	59.74	37.67	1.91	64.78
$\beta = 1$	4.58	43.27	47.74	116.83	61.05	55.78	33.60	0.53	20.98
L($\beta = 0$)	6.60	48.09	59.71	116.63	46.99	69.64	45.69	4.92	147.46
				Infinite Average Maturity					
$\beta = -1$	9.75	36.23	77.44	130.43	29.43	101.00	52.03	11.20	165.33
$\beta = -0.5$	9.20	42.97	74.65	126.85	32.16	94.69	52.06	10.13	171.30
$\beta = 0.5$	7.42	45.72	65.84	123.29	42.12	81.17	45.63	4.62	114.32
$\beta = 1$	4.98	36.19	50.73	120.95	59.60	61.35	33.52	0.39	12.36
L($\beta = 0$)	8.38	45.37	70.58	124.43	36.61	87.82	49.53	7.69	153.83

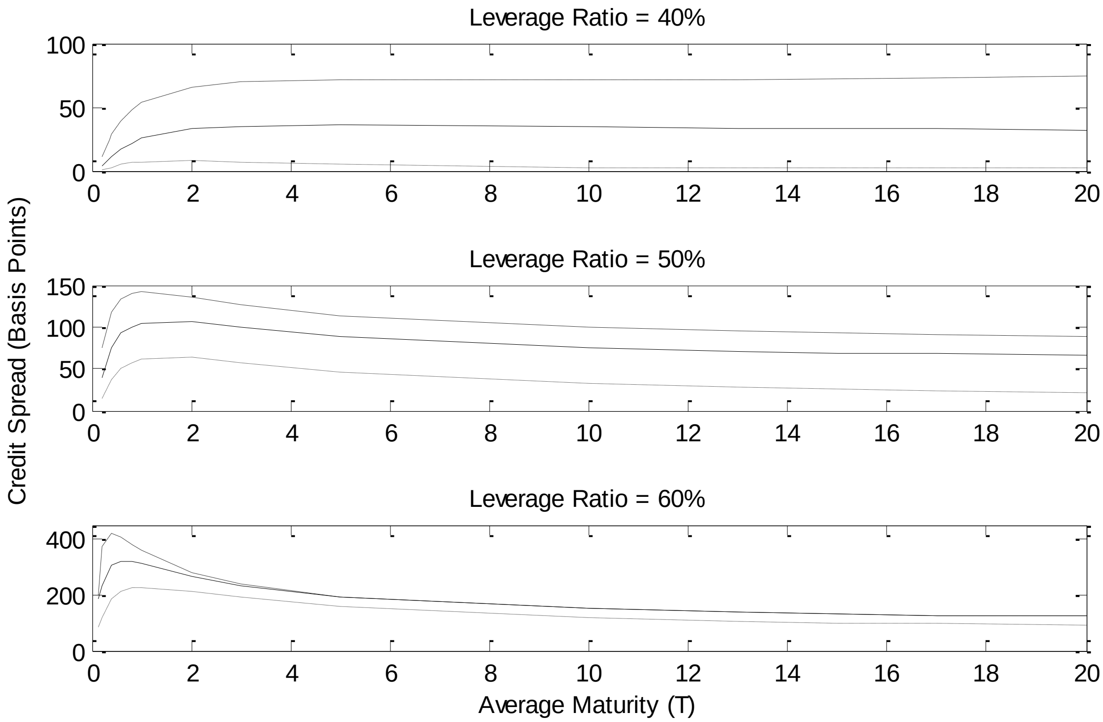

Figure 4: The term structure of credit spreads under different leverage ratio. This figure depicts the term structure of credit spreads for the Leland (1994b) model (solid line) and CEV structural models with $\beta = -1$ (dashed lines), and $\beta = 1$ (dotted lines) under 40% (Top), 50% (Middle) and 60% (Bottom) leverage ratios. It is assumed that current asset value $V = 100$ dollars, risk free rate $r = 0.08$, the firm's payout rate $q = 0.06$, tax rate $w = 0.35$, and proportional bankruptcy cost $\alpha = 0.5$. The coupon rate is determined by making the debt issued at par value under the endogenous default boundary. The initial volatilities are the same for these scenarios and $\sigma_0 = 20\%$. For the given β under the CEV diffusion process, $\theta = \sigma_0 \;\; V^{\beta}$.

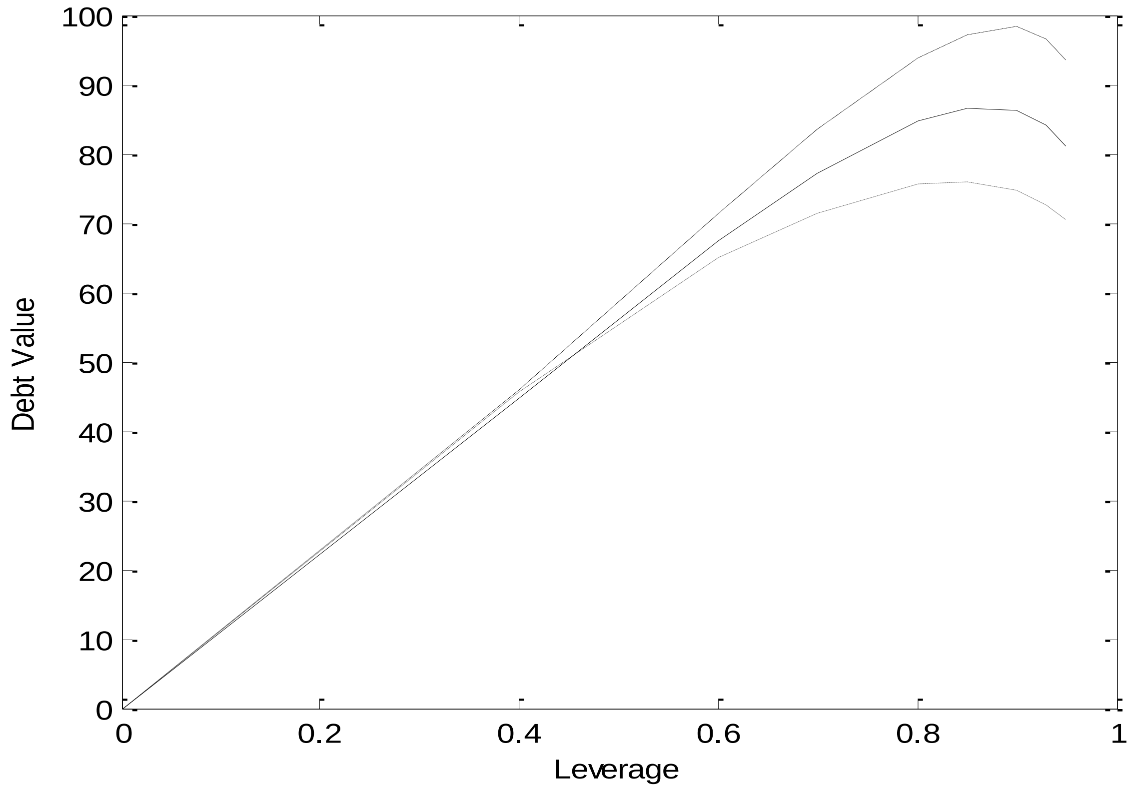

Figure 5: Debt value as a function of leverage. This figure depicts the value of debt for different leverage ratios under the Leland(1994b) model(solid line), CEV structural model with $\beta = -1$ (dashed line), and $\beta = 1$ (dotted line). It is assumed that current asset value $V = 100$ dollars, risk free rate $r = 0.08$, the firm's payout rate $q = 0.06$, tax rate $w = 0.35$, and proportional bankruptcy cost $\alpha = 0.5$. The coupon rate is determined by making the debt issued at par value under the endogenous default boundary. The initial volatilities are the same for these scenarios and $\sigma_0 = 20\%$. For the given β under the CEV diffusion process, $\theta = \sigma_0 \ V^{\beta}$. The average maturity of debt is assumed to be 5 years.

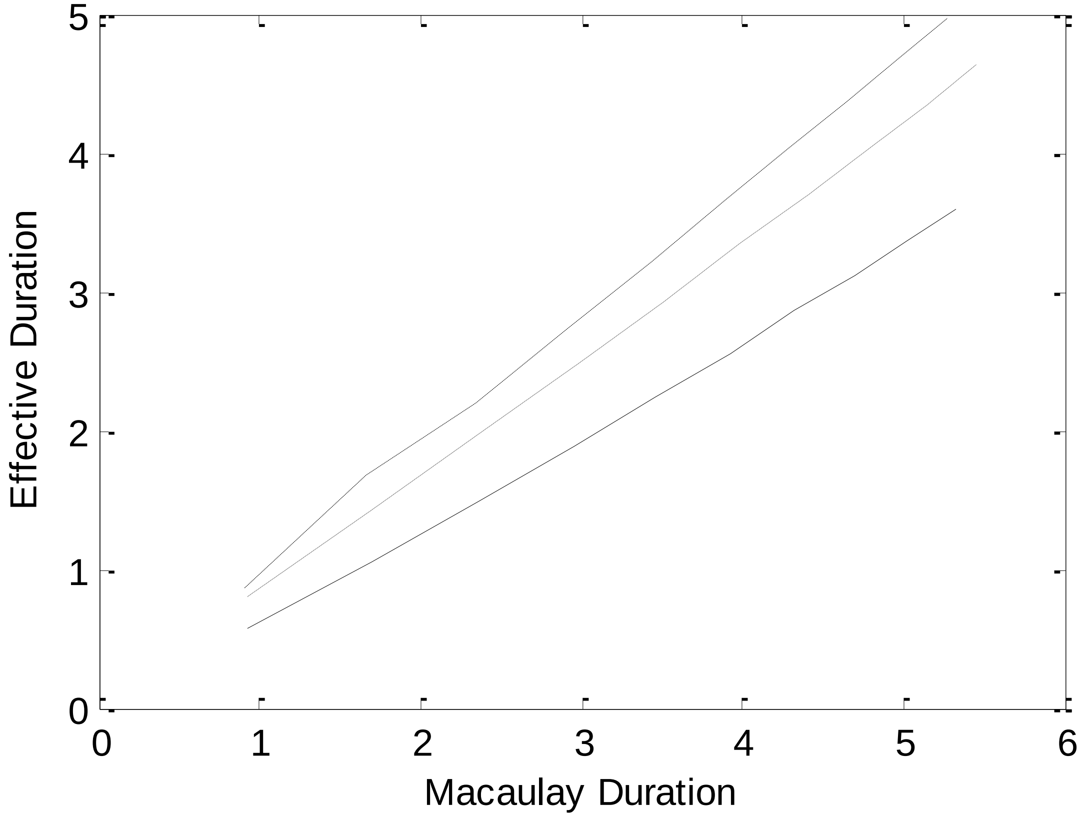

Figure 6: Effective duration with respect to Macaulay duration. This figure depicts the change of effective duration of bonds with respect to their Macaulay Duration for the Leland (1994b) model (solid line)and CEV structural models with $\beta = -1$(dashed line) and $\beta = 1$(dotted line). It is assumed that current asset value $V = 100$ dollars, risk free rate $r = 8\%$, the firm's payout rate $q = 0.06$, tax rate $w = 0.35$, and proportional bankruptcy cost $\alpha = 0.5$. The coupon rate is determined by making the debt issued at par value under the endogenous default boundary. The leverage ratio is assumed to be 50%. The initial volatilities are the same for these scenarios and $\sigma_0 = 20\%$. For the given β under the CEV diffusion process, $\theta = \sigma_0 \ V^{\beta}$.

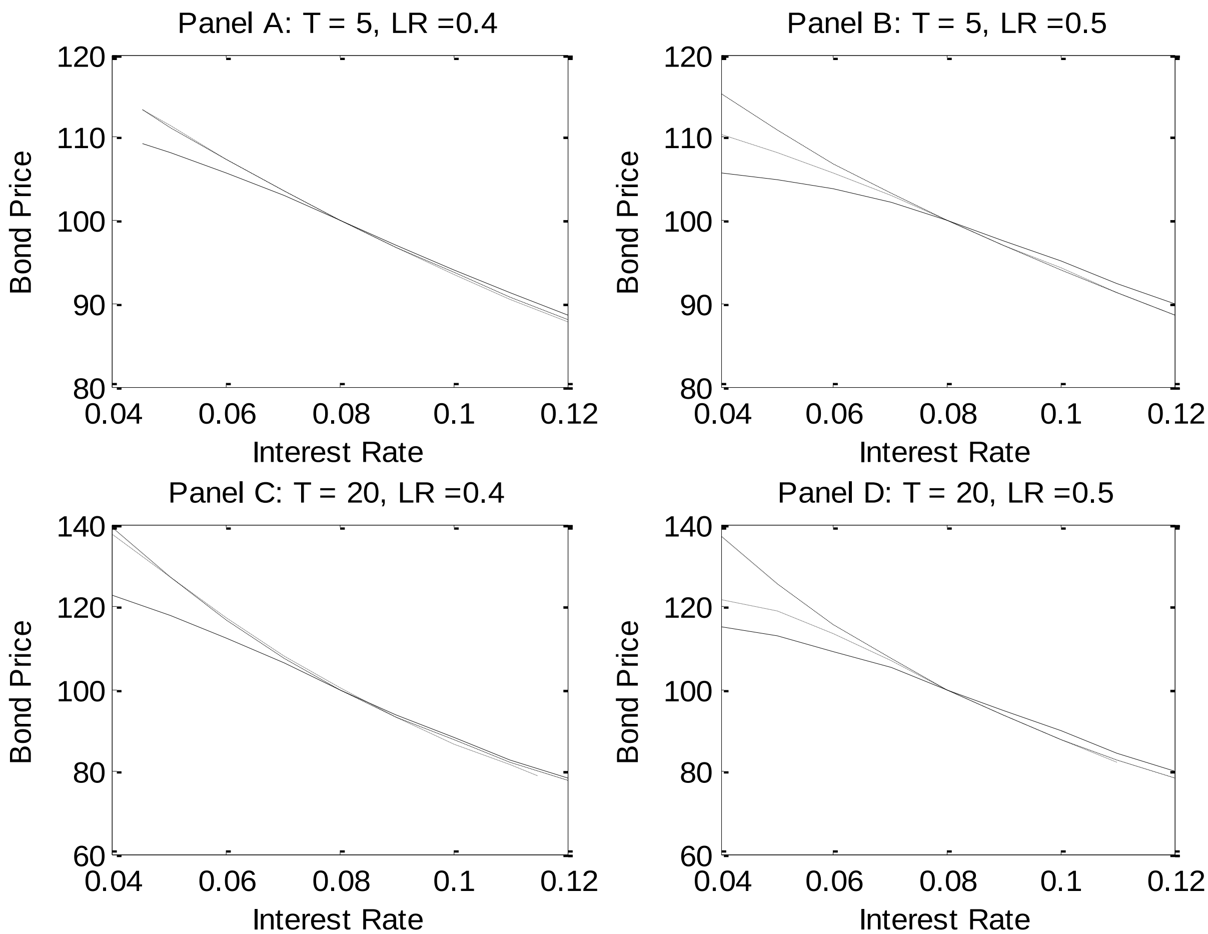

Figure 7: Bond price as a function of the risk-free interest rate. The graphs depict the bond price per \$100 face value as a function of risk-free rate of interest for the Leland (1994b) structural model (solid lines) and CEV structural models with $\beta = -1$(dashed lines) and $\beta = 1$(dotted lines). Panel A shows the bonds with 5-year average maturity and 40% leverage ratio; Panel B shows the bonds with 5-year average maturity and 50% leverage ratio; Panel C shows the bonds with 20-year average maturity with 40% leverage ratio; and Panel D shows the bonds with 20-year average maturity with 50% leverage ratio. It is assumed that current asset value $V = 100$ dollars, the firm's payout rate $q = 0.06$, tax rate $w = 0.35$, and proportional bankruptcy cost $\alpha = 0.5$. The coupon rate is determined by making the debt issued at par value under the endogenous default boundary when the interest rate is 8%. The initial volatilities are the same for these scenarios and $\sigma_0 = 20\%$. For the given β under the CEV diffusion process $\theta = \sigma_0 \ V^{\beta}$.

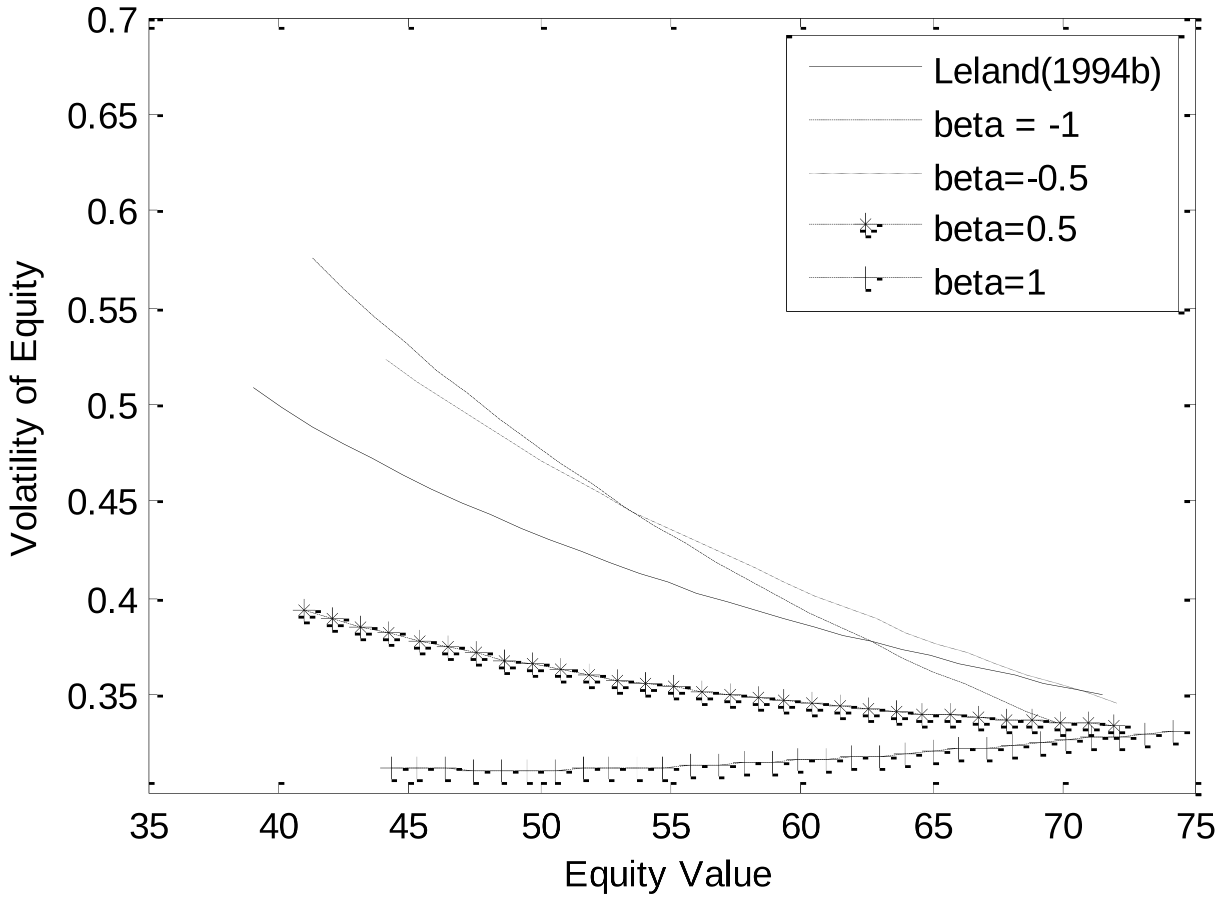

Figure 8: Volatility of equity with respect to the level of equity value. This figure depicts the volatilities of equity under the Leland (1994b) model (solid line) and CEV structural models with $\beta = -1 \beta = -1$ (dashed line), $\beta = -0.5$ (dotted line), $\beta = 0.5$ (star-dashed line) and $\beta = 1$ (plus-dashed line). It is assumed that current asset value $V = 100$ dollars, risk free rate $r = 0.08$, the firm's payout rate $q = 0.06$, tax rate $w = 0.35$, and proportional bankruptcy cost $\alpha = 0.5$. The average maturity of debt is 5 years. The leverage ratio is the optimal leverage ratio computed under the endogenous default boundary. The coupon rate is determined by making the debt issued at par value under the ndogenous default boundary. The initial volatilities are the same for these scenarios and $\sigma_0 = 20\%$. For the given β under the CEV diffusion process $\theta = \sigma_0 / V^{\beta}$.

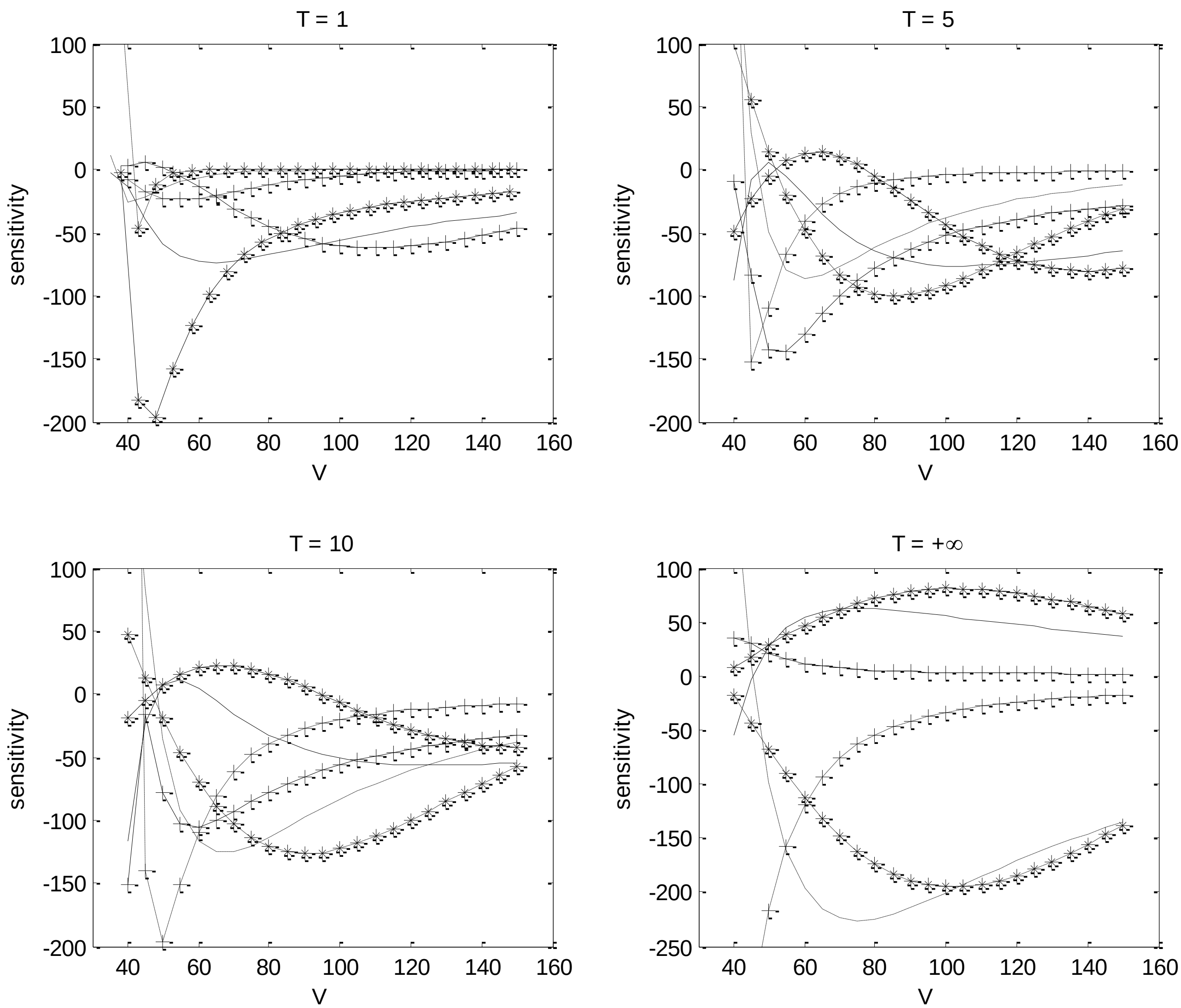

Figure 9: Sensitivity of equity and debt values to total asset risk. This figure depicts the sensitivity of equity value and debt value to total asset risk measured by the volatility of assets, for 1-year, 5-year, 10-year and perpetual bonds, shown under the L model with solid (dashed) lines for the sensitivity of equity (debt), under the CEV structural model with $\beta = 1$ with plus-solid (plus-dashed) lines for the sensitivity of equity (debt), and the CEV model with $\beta = -1$ with star-solid (star-dashed) lines for the sensitivity of equity (debt). The total coupon payment and face value of debt are determined such that the capital structure is optimal for firm value $V = 100$. The particular values for each bond refer to Table 1. It is assumed that the risk free rate $r = 0.08$, the firm's payout rate $q = 0.06$, tax rate $w = 0.35$, and proportional bankruptcy cost $\alpha = 0.5$.

Table 2: Calibration of Model Parameters

Panel A				
	Aa	A	Baa	Ba
Average leverage(D/v)	31.6%	41.7%	44.8%	49.8%
Payout Rate	6%	6%	6%	6%
Risk Free Rate	8%	8%	8%	8%
Recovery Rate	50%	50%	50%	50%
Tax Rate	0.35	0.35	0.35	0.35
Risk Premium	4%	4%	4%	4%
Panel B				
V_0	100	100	100	100
K	31.6	41.7	44.8	49.8
Average Implied volatility(1983-2008)	18.97%	17.69%	19.11%	23.36%
Average Implied volatility(1983-2010)	19.33%	18.30%	19.50%	24.21%

Table 3: CEV structural model parameter estimation by fitting Moody's historical CDPs σ_M^I

This table reports the estimation of β and σ_0 in the CEV structural model by minimizing the sum of absolute deviations from Moody's historical CDPs for terms from 1-year to 10-years.

$$\underset{\beta,\theta}{MIN} \sum \left| \sigma_{CEV}^I - \sigma_M^I \right|$$

The average leverage, payout rate, risk free rate, recovery rate, tax rate and risk premium are assumed to be the same as in Panel A in Table 2 for different rating categories of debt. We assume the initial asset value is 100 and the exogenous default boundary equals the face value of the debt. $\overline{\sigma}_M^I$ is the average implied volatility by the LT model.

	Period of 1983-2008				Period of 1983-2010			
Rating	β	σ_0	$\sum\left\|\sigma_{CEV}^I - \sigma_M^I\right\|$	$\sum\left\|\sigma_M^I - \overline{\sigma}_M^I\right\|$	β	σ_0	$\sum\left\|\sigma_{CEV}^I - \sigma_M^I\right\|$	$\sum\left\|\sigma_M^I - \overline{\sigma}_M^I\right\|$
Aa	-5	6.5%	0.0498	0.3146	-6	6.3%	0.0530	0.3128
A	-4	8%	0.0621	0.1832	-5	7.6%	0.0778	0.2036
Baa	-4	9.5%	0.0747	0.1914	-4	9.8%	0.0836	0.1866
Ba	-2	16%	0.0884	0.1208	-2	16.5%	0.1010	0.1075

Table 4: Characteristics of firms under different models with Jumps

This table reports the characteristics of firms for Leland's model (L), Leland's model with jump (L-J), CEV structural model (CEV) and CEV structural model with jump (CEV-J). The bonds pays continuous coupon and the coupon payment equals the continuous coupon in Table 1 for each scenario. The default events only occur on the semi-annual basis and the default boundary is equal to the endogenous default boundary of the optimally levered firm in Table 1 for each scenario. The intensity of the jump is $\eta_J = 1/10$. The jump amplitude for L-J follows a log-normal distribution and for CEV-J follows a binomial distribution with $\mu_J = -0.05, \sigma_J = 0.2$. It is assumed that current asset value $V = 100$ dollars, the firm's payout rate $q = 0.06$, tax rate $w = 0.35$, and proportional bankruptcy cost $\alpha = 0.5$. The initial volatilities are the same for these scenarios and $\sigma_0 = 20\%$. For each given β under the CEV diffusion process, $\theta = \sigma_0 / V^{\beta}$.

	Models	Leverage (Percent)	Firm Value (Dollars)	Equity Value (Dollars)	Total Debt Value (Dollars)	Credit Spread (Basis Points)
1-year maturity bond						
L($\beta = 0$)	L	0.2829	107.63	77.18	30.45	1.29
	L-J	0.2849	106.87	76.42	30.44	1.39
$\beta = -1$	CEV	0.3867	109.92	67.42	42.50	44.66
	CEV-J	0.3907	108.73	66.24	42.49	44.99
$\beta = -0.5$	CEV	0.3263	107.93	72.71	35.22	12.09
	CEV-J	0.3293	106.92	71.71	35.21	12.26
$\beta = 0.5$	CEV	0.2709	108.58	79.16	29.42	0
	CEV-J	0.2737	107.48	78.07	29.41	0
$\beta = 1$	CEV	0.2923	108.76	76.96	31.80	0
	CEV-J	0.2965	107.22	75.43	31.79	0
5-year maturity bond						
L($\beta = 0$)	L	0.5125	114.58	55.86	58.72	90
	L-J	0.5154	113.11	54.81	58.30	97
$\beta = -1$	CEV	0.6718	123.00	40.38	82.63	253
	CEV-J	0.6792	120.42	38.63	81.79	264
$\beta = -0.5$	CEV	0.6094	118.06	46.12	71.95	188
	CEV-J	0.6155	115.79	44.52	71.28	198
$\beta = 0.5$	CEV	0.4255	113.62	65.27	48.35	25
	CEV-J	0.4299	111.95	63.82	48.13	29
$\beta = 1$	CEV	0.4064	114.83	68.15	46.67	5.6
	CEV-J	0.4117	113.02	66.49	46.53	8
10-year maturity bond						
L($\beta = 0$)	L	0.5984	118.61	47.64	70.97	130
	L-J	0.5989	116.90	46.88	70.02	143
$\beta = -1$	CEV	0.7112	126.52	36.54	89.98	210
	CEV-J	0.7138	124.04	35.50	88.54	227
$\beta = -0.5$	CEV	0.6683	122.05	40.49	81.56	189
	CEV-J	0.6706	119.65	39.41	80.24	206
$\beta = 0.5$	CEV	0.5153	116.99	56.70	60.29	58
	CEV-J	0.5180	114.94	55.40	59.54	68
$\beta = 1$	CEV	0.4760	117.57	61.60	55.96	18
	CEV-J	0.4802	115.34	59.95	55.39	27

Table 5: Characteristics of firms under different models with jumps

This table reports the characteristics of firms for Leland's model with jump (L-J) and CEV structural model with jump (CEV-J) under varying jump calibrations. The 5-year bond pays continuous coupon and the coupon payment equals the continuous coupon in Table 1 for each scenario. The default events only occur on a semi-annual basis and the default boundary is equal to the endogenous default boundary of the optimally levered firm in Table 1 for each scenario. For the base case, the intensity of the jump is $\eta_J = 1/10$. The jump amplitude for L-J follows a log-normal distribution, and for CEV-J follows a binomial distribution with $\mu_J = -0.05, \sigma_J = 0.2$. It is assumed that current asset value $V = 100$ dollars, the firm's payout rate $q = 0.06$, tax rate $w = 0.35$, and proportional bankruptcy cost $\alpha = 0.5$. The initial volatilities are the same for these scenarios and $\sigma_0 = 20\%$. For each given β under the CEV diffusion process, $\theta = \sigma_0 / V^{\beta}$.

Models	Leverage (Percent)	Firm Value (Dollars)	Equity Value (Dollars)	Total Debt Value (Dollars)	Credit Spread (Basis Points)
			L-J($\beta = 0$)		
Base Case	0.5154	113.11	54.81	58.30	97
$\eta_J = 1/5$	0.5182	111.71	53.82	57.89	103
$\eta_J = 1/20$	0.5140	113.84	55.33	58.51	94
$\sigma_J = 0.25$	0.5156	112.87	54.68	58.19	99
$\sigma_J = 0.15$	0.5153	113.33	54.93	58.40	95
$\mu_J = -0.1$	0.5178	112.13	54.07	58.07	101
$\mu_J = 0.05$	0.5109	114.84	56.17	58.67	91
				$\beta = -1$	
Base Case	0.6792	120.42	38.63	81.79	264
$\eta_J = 1/5$	0.6866	117.87	36.94	80.93	275
$\eta_J = 1/20$	0.6755	121.71	39.50	82.21	258
$\sigma_J = 0.25$	0.6799	120.09	38.44	81.64	266
$\sigma_J = 0.15$	0.6787	120.68	38.78	81.90	262
$\mu_J = -0.1$	0.6858	118.33	37.18	81.15	272
$\mu_J = 0.05$	0.6681	124.11	41.19	82.92	249
				$\beta = 1$	
Base Case	0.4117	113.02	66.49	46.53	8
$\eta_J = 1/5$	0.4165	111.34	64.97	46.37	11
$\eta_J = 1/20$	0.4093	113.82	67.23	46.59	7
$\sigma_J = 0.25$	0.4126	112.66	66.18	46.49	8.8
$\sigma_J = 0.15$	0.4110	113.29	66.73	46.56	7.5
$\mu_J = -0.1$	0.4152	111.87	65.43	46.45	9.5

$\mu_J = 0.05$	0.4065	114.72	68.09	46.64	6.2

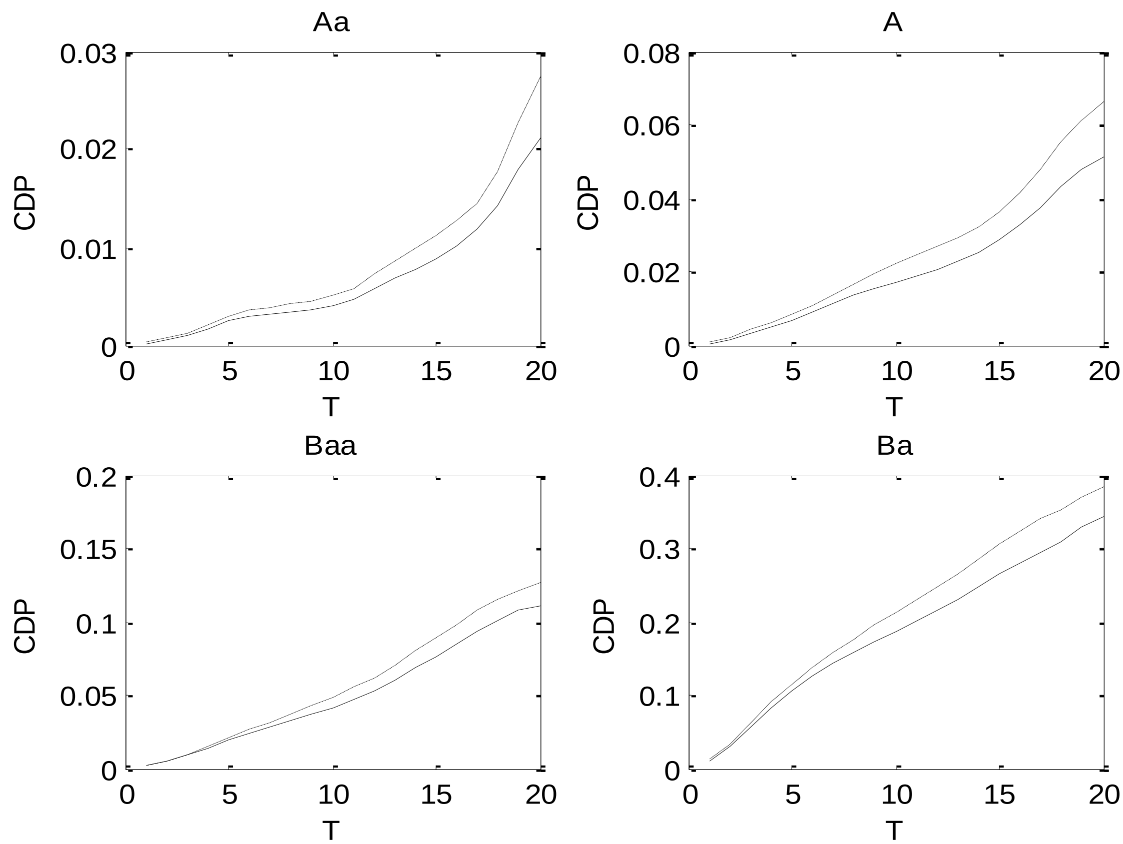

Figure 10: Cumulative Default Probabilities of Aa, A, Baa and Ba rated corporate bonds. The dashed lines show the cumulative default probability (CDP) of corporate debt during 1983-2010. The solid lines show the cumulative default probabilities of corporate debt during 1983-2008.

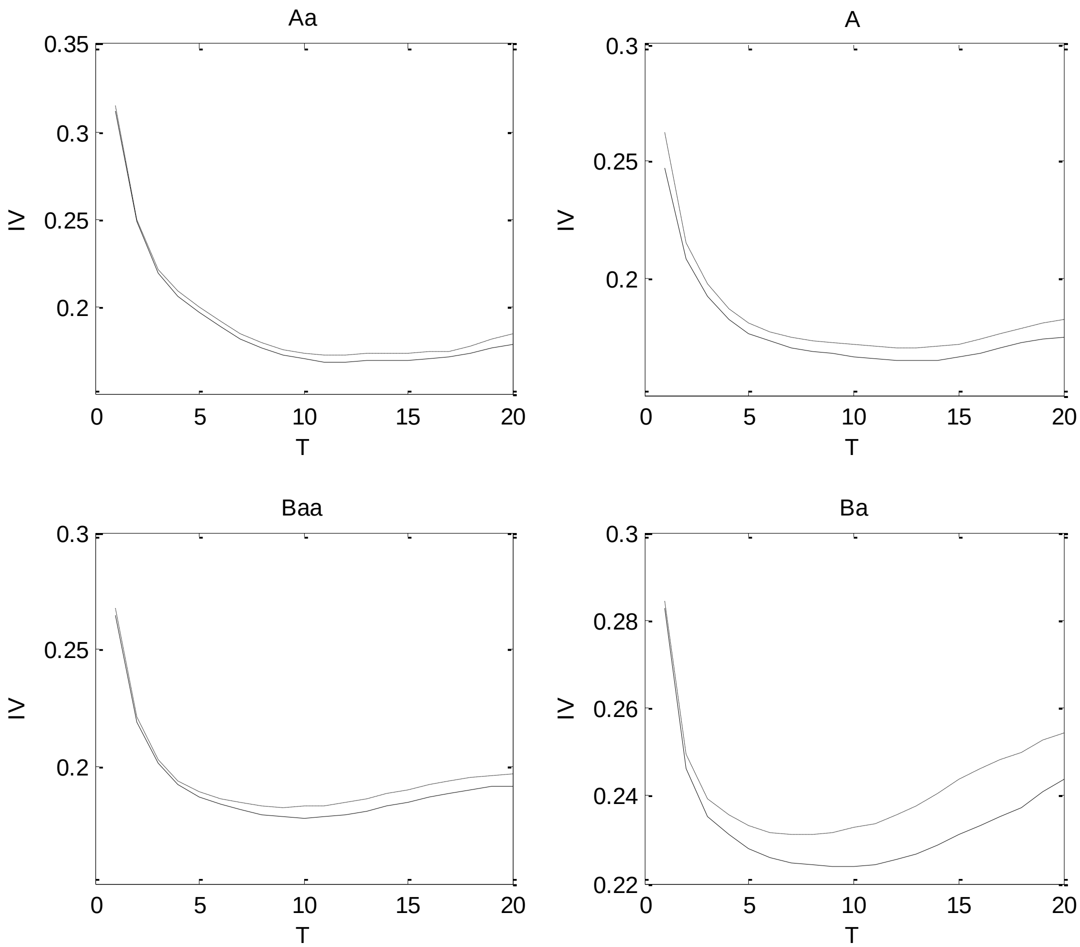

Figure 11: Implied volatility of historical cumulative default probabilities. The dashed lines show the implied volatilities of the cumulative default probabilities during 1983-2010 by LT model, while the solid lines show the implied volatilities during 1983-2008. The initial asset value equals 100 and the debt value is chosen from the empirical leverage ratio for the different ratings. The coupon is calculated by making the debt issued at par value. Tax rate is 35% and recovery rate is 50% for all the debts. The average maturity of debt is 10 years. The exogenous default boundaries are equal to the value of debt for each scenario.

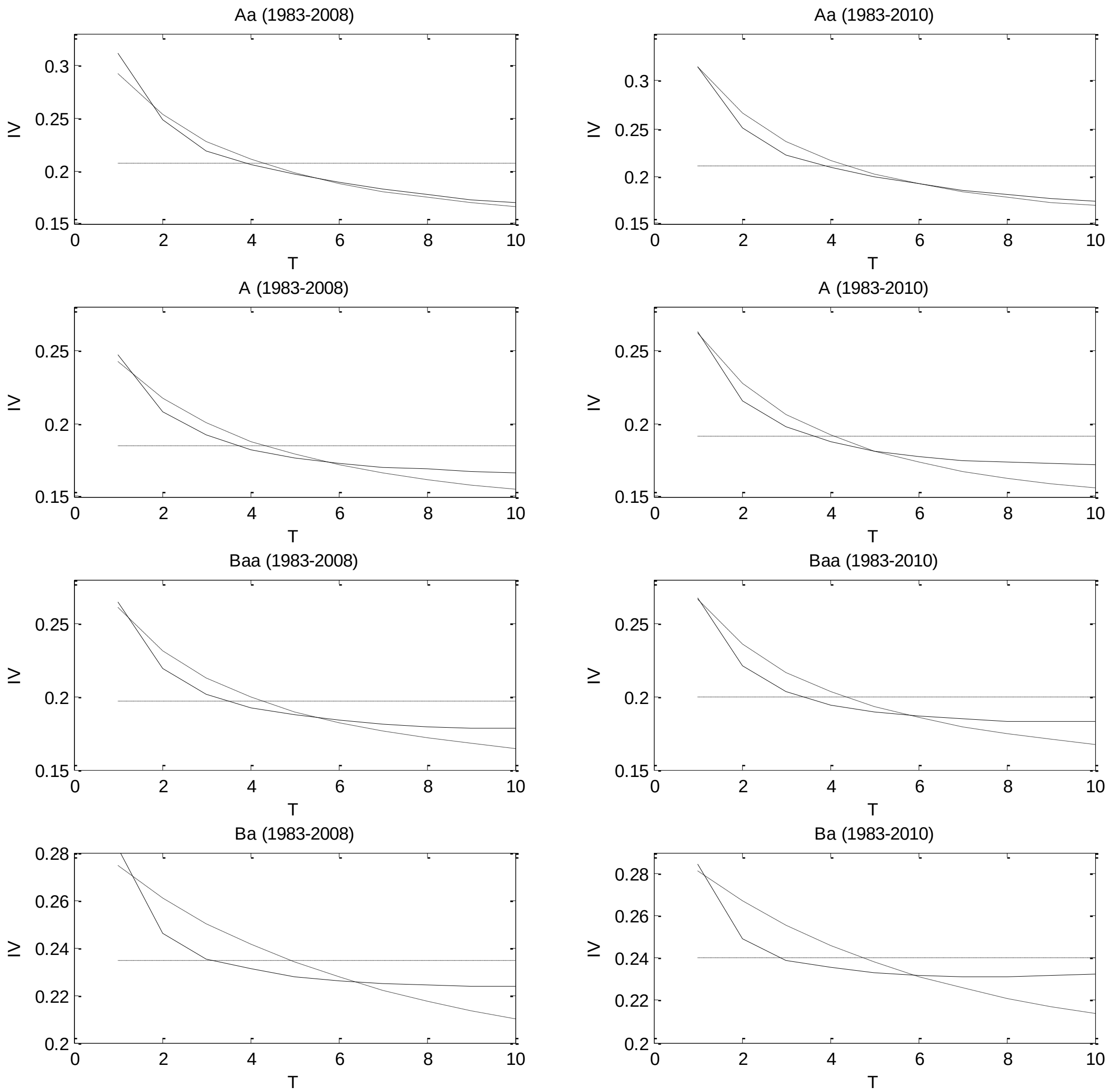

Figure 12: Term structures of Implied Volatilities (IV) of CEV structure model. This figure depicts the term structure of IV for Moody's historical CDPs (solid lines) and CEV structural model (dashed lines). The LT model with exogenous default boundary is used to calculate the IVs. The average IVs for Moody's historical CDPs are shown by dashed-dot lines for Aa, A, Ba, B rated debt during the periods 1983-2008 and 1983-2010. The average leverage, payout rate, risk free rate, recovery rate, tax rate and risk premium are assumed to be the same as in Panel A in Table 2 for different rating categories of debt. We assume the initial asset value is 100 and the exogenous default boundary equals the face value of the debt. The values of β and σ_0 in CEV structural model are calculated by minimizing the sum of absolute deviations from Moody's historical CDPs for terms from 1-year to 10-years and shown in Table 5.

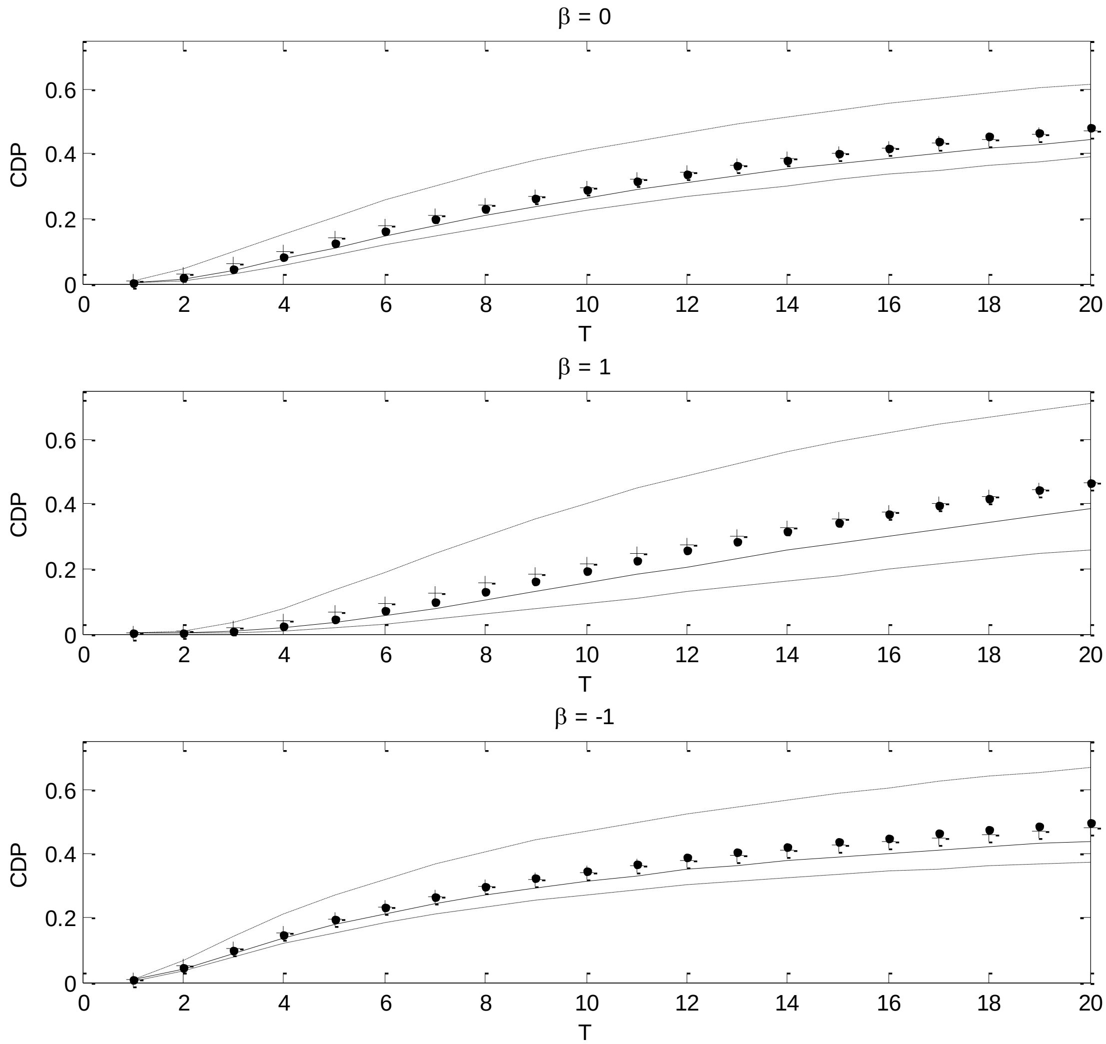

Figure 13: CDPs for different models with jumps. This figure depicts the CDPs for diffusion-jump model ($\beta = 0$) and CEV-jump models ($\beta = 1, \beta = -1$) under varying calibrations of jump parameters. The dashed lines show the diffusion model and CEV model without jumps. The solid lines show the base case for the diffusion-jump and CEV-jump models. For the base case of jump calibration, $\mu_J = -0.05, \sigma_J = 0.2$ and $\eta_J = 1/10$. The dashed-dot lines show the case of $\eta_J = 1/2$; the dot lines show the case of $\mu_J = -0.1$; the plus lines show the case of $\sigma_J = 0.4$ with the other parameters same as in the base case. The initial asset value, $V_0 = \$100$, risk free rate, $r = 0.08$, payout ratio, $q = 0.06$ and initial asset volatility $\sigma_0 = 20\%$. The exogenous default boundary equals $50 for all the scenarios.

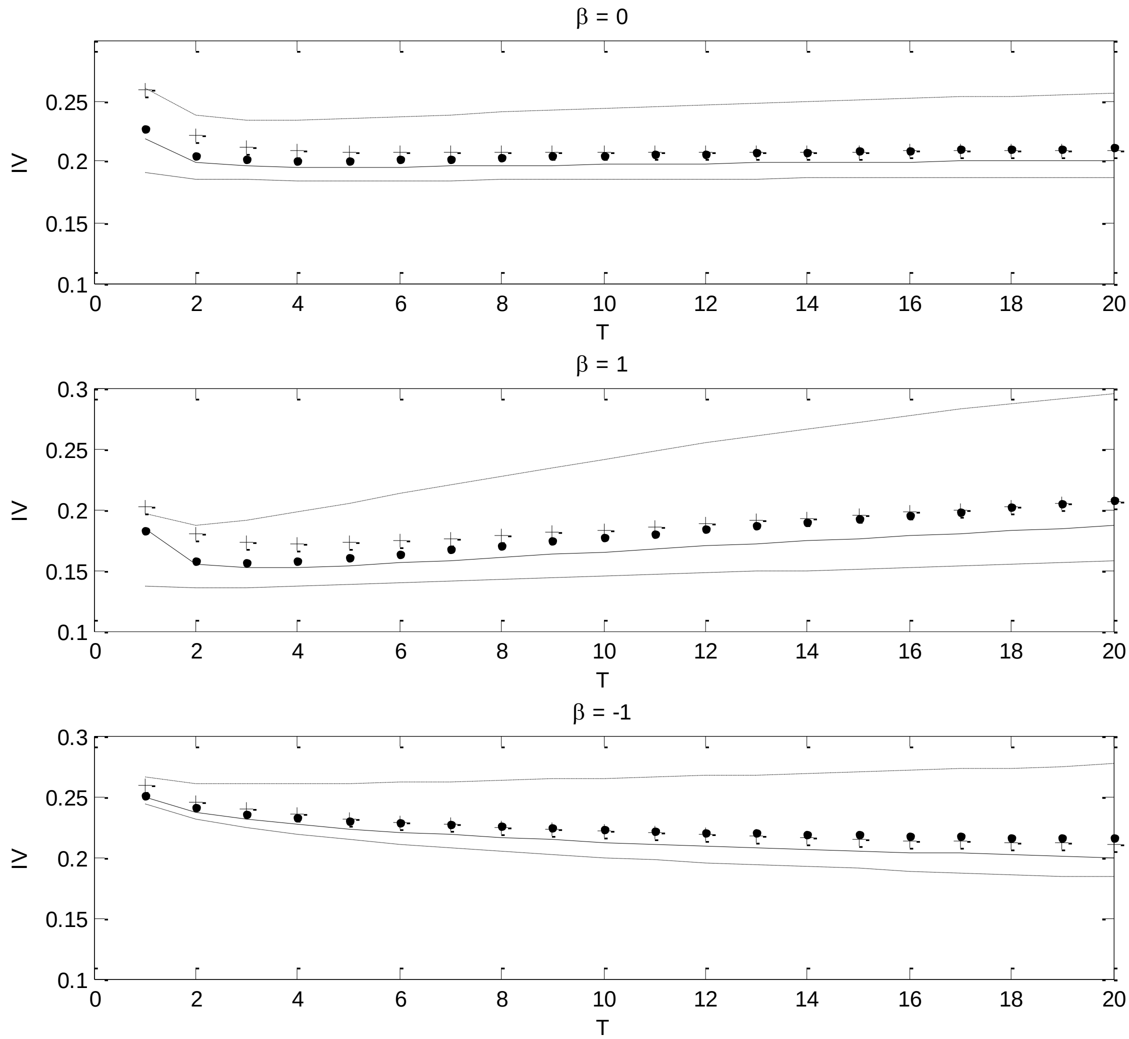

Figure 14: Term structure of IVs for different models with jumps. This figure depicts the term structure of IVs for diffusion-jump model ($\beta = 0$) and CEV-jump models ($\beta = 1, \beta = -1$) under varying calibrations of jump parameters. The dashed lines show the diffusion model and CEV model without jumps. The solid lines show the base case for diffusion-jump and CEV-jump models. For the base case of jump calibration, $\mu_J = -0.05, \sigma_J = 0.2$ and $\eta_J = 1/10$. The dashed-dot lines show the case of $\eta_J = 1/2$; the dot lines show the case of $\mu_J = -0.1$; the plus lines show the case of $\sigma_J = 0.4$ with the other parameters same as in the base case. The initial asset value, $V_0 = \$100$, risk free rate, $r = 0.08$, payout ratio, $q = 0.06$ and initial asset volatility $A(t) = \int_{\tau=0}^{\tau=t} f(\tau) d\tau$. The exogenous default boundary equals \$50 for all the scenarios.