Stochastic Dominance and Option Pricing in Discrete and Continuous Time: an Alternative Paradigm

By Ioan Mihai Oancea[1] and Stylianos Perrakis[2]

Abstract

This paper examines option pricing in a universe in which it is assumed that markets are incomplete. It derives multiperiod discrete time option bounds based on stochastic dominance considerations for a risk-averse investor holding only the underlying asset, the riskless asset and (possibly) the option for any type of underlying asset distribution, discrete or continuous. It then considers the limit behavior of these bounds for special categories of such distributions as trading becomes progressively more dense, tending to continuous time. It is shown that these bounds nest as special cases most, if not all, existing arbitrage- and equilibrium-based option pricing models. Thus, when the underlying asset follows a generalized diffusion both bounds converge to a single value. For jump-diffusion processes, stochastic volatility models, and GARCH processes the bounds remain distinct and define several new option pricing results containing as special cases the arbitrage-based results.

Key words: Option pricing, option bounds, incomplete markets, jump-diffusion processes, stochastic volatility, GARCH processes.

[1] Director, LDI Solutions, National Bank of Canada, 130 King Street West, Toronto, Ont., M5X 1J9, Canada, phone: (416) 869-6752, ioan.oancea@tres.bnc.ca.

[2]John Molson School of Business, Concordia University, 1455 de Maisonneuve Blvd. W, Montreal, QC H3G 1M8, Canada, Phone: (+1-514) 848-2424 ext. 2963, Fax: (+1-514) 848-4500, sperrakis@jmsb.concordia.ca. Perrakis acknowledges the support of SSHRC, the Social Sciences and Humanities Research Council of Canada.

1. Introduction

There have now been more than thirty years since the publication of the seminal Black and Scholes (1973) and Merton (1973) studies that established the foundations of modern option pricing theory. The significance of these studies lay as much in the arbitrage valuation methodology that they introduced as in the derived results. The option prices in these studies were derived by the construction of a riskless hedge that was supposed to earn a return equal to the riskless rate of interest. Alternatively, they may be derived by the construction of a portfolio containing the underlying and the riskless assets that replicates the option perfectly in all states of the world. If such a replication is feasible then it can be shown that the price of any contingent claim on the underlying asset is equal to the discounted expected payoff of the claim under a risk neutral transformation of the underlying asset's return distribution, generally denoted as the Q-distribution.[3] It is this arbitrage-based methodology that underlies the overwhelming majority of derivatives pricing studies in modern financial theory.

The arbitrage methodology relies on two fundamental assumptions that cannot be relaxed easily in most applications. These are known as dynamic market completeness and frictionless trading. The former is most often identified with a distribution of the returns of the underlying asset that follows a univariate diffusion process. The latter is generally interpreted as the absence of transaction costs in trading the underlying asset. While there are extensions of the basic arbitrage methodology that can take care of several forms of market incompleteness, there is no satisfactory arbitrage-based approach to option pricing in the presence of transaction costs.[4] Even for the cases of market incompleteness, the proposed option pricing models are specific to each type of underlying distribution, without any unifying theory; they thus require the identification of the type of underlying asset distribution before developing the method for pricing the option.[5]

In this paper we examine an alternative class of option pricing models that have existed for several years but have received little attention in mainstream financial research.[6] We claim that the approach introduced by this class of models constitutes a new option pricing paradigm, distinct and more general than arbitrage. It can produce general solutions to problems that cannot be solved by arbitrage, while being simultaneously

[3] See Harrison and Kreps (1979) and Harrison and Pliska (1981).

[4] See Merton (1989), and Soner et al (1996).

[5] Mixed jump-diffusion processes, GARCH processes and stochastic volatility are three examples of market incompleteness that have appeared in the literature. In such incompleteness cases the Q-distribution is not uniquely defined by arbitrage methods alone and additional assumptions are needed to derive it. See Merton (1976), Hull and White (1987), Amin and Ng (1993), Amin (1993), and Duan (1995) for some examples.

[6] This class of models was introduced by Perrakis and Ryan (1984) and extended by Levy (1985) and Ritchken (1985) in a single period context. The multiperiod extension was done in Perrakis (1986, 1988) and Ritchken and Kuo (1988).

capable of producing the same results as arbitrage whenever the latter works. These problems concern primarily market incompleteness, but the stochastic dominance approach has recently been extended to incorporate frictions such as proportional transaction costs in trading the underlying asset.[7] Under their existing form the stochastic dominance models are applicable to underlying assets whose discrete time returns distributions are independent and identically distributed (iid) between successive time periods. Unlike the riskless hedge of the mainstream arbitrage approach, these models produce price bounds whose violation triggers stochastically dominating strategies involving portfolios containing the option, the underlying asset and the riskless asset.

In this paper we first extend the stochastic dominance models to underlying asset distributions whose returns are Markovian but non-iid. We then show that at least for "plain vanilla" call and put options the results of this stochastic dominance option pricing approach contain as special cases virtually the entire set of results produced by the conventional arbitrage methodology. Furthermore, these results are common to all types of underlying asset distributions, and not only to distributions belonging to a given class. They can thus be used to price options on assets for which the return distribution is known only under the form of a histogram extracted from historically observed values, or where such a distribution has been distorted to incorporate subjective forecasts. Last but not least, since the models' results are formulated in discrete time and cover continuous time as a limit, they may thus be used to value American, as well as European options.

The results of this stochastic dominance approach to option pricing take the form of two bounds on admissible option prices; an observed market price outside the bounds triggers an expected utility-improving strategy of writing or purchasing the mispriced option. We show that for the basic case of a dynamically complete market without transaction costs, in which the underlying asset follows a generalized Ito process, the two bounds of the stochastic dominance approach converge to a single value, the one corresponding to the arbitrage-derived option price, under all circumstances.[8] This convergence takes place even though the market for the underlying asset is incomplete for any discrete time subdivision of the time to expiration of the option. Since the convergence takes place even when there is no closed-form option price derived from the arbitrage-based approach, the two bounds provide an alternative method of deriving such a price via Monte Carlo simulations of the general form of the equations of the bounds.

Next we examine the stochastic dominance approach in cases in which the market is dynamically incomplete. We examine the two main types of market incompleteness that have appeared in the literature, a mixed jump-diffusion process for the distribution of the underlying asset, and a stochastic volatility process for this same distribution. For the mixed jump-diffusion process there are two possible approaches in the earlier studies. Merton (1976) assumes that the risk arising out of the jump process is fully diversifiable and that the amplitude of the jumps is lognormally distributed. He then applies the

---

[7]See Constantinides and Perrakis (2002, 2007). For an empirical application of the stochastic dominance approach in the presence of frictions see Constatinides *et al* (2007).

[8]This covers the simple Black-Scholes diffusion as well as several other models such as, for instance, the so-called constant elasticity of variance (CEV) approach, as in Cox and Rubinstein (1985, pp. 361-364).

traditional arbitrage method to derive an option price that is an expectation of Black-Scholes-type expressions. Other studies such as Bates (1991), Amin (1993) and Amin and Ng (1993), use general equilibrium-type arguments, in which the marginal utility of consumption of a representative investor plays the role of a pricing operator;[9] the option price becomes then a function of the investor's risk aversion parameter. On the other hand, the stochastic volatility models handle market incompleteness by specifying, somewhat arbitrarily, a "price" of the volatility risk in order to derive a risk neutral process, a form of the Q-distribution that can be used to price contingent claims.[10]

We show in this paper that under mixed jump-diffusion processes the bounds of the stochastic dominance approach converge to two distinct values for any distribution of the jump amplitude. These values are expectations of the option payoff under two different risk neutral Q-distributions, identified by the solutions of two partial differential equations (pde). Further, these bounds contain all the values derived by the equilibrium models, including the one corresponding to the Merton (1976) assumption, as well as those of Bates (1991) and Amin (1993). As for the stochastic volatility models, we show that the stochastic dominance bounds can be suitably redefined to cover virtually all the stochastic volatility models that have appeared in the literature in which the volatility risk is assumed fully diversifiable, as in Hull and White (1987); in such a case the two bounds converge to a single value. By contrast, only partial results are available by stochastic dominance when the volatility risk is priced, results applicable only to investors with constant proportional risk aversion (CPRA) utility functions. For such investors we identify bounds on option prices that are applicable to particular subsets of the risk aversion parameter value. Last, we show that under GARCH processes the stochastic dominance bounds, although they no longer define reservation buy/write prices for options, still define a nonempty interval of admissible option prices and can be used for pricing options.

In the next section we summarize briefly the option bounds derived by the stochastic dominance approach as it has appeared in the literature till now, when the returns of the underlying asset are independent and identically distributed (iid). The extension to a Markovian structure of returns is discussed in section III, while the limiting forms of the bounds when the stock returns follow a general Ito process are shown in section IV. Sections V, VI and VII present the results for mixed jump-diffusions, stochastic volatility and GARCH processes. Section VIII concludes.

2. The Stochastic Dominance Approach

The stochastic dominance results were initially derived in Perrakis and Ryan (1984) by eliminating stochastically dominating strategies in comparing two portfolios. They were extended by considering a single period market equilibrium model in Ritchken (1985), and by second order stochastic dominance (SSD) comparisons of the terminal wealth

---

[9]A similar equilibrium argument underlies the GARCH option pricing models, first presented in Duan (1995).

[10] Hull and White (1987) assume that the stochastic volatility risk is diversifiable (the price of volatility risk is zero); while Heston (1993) and Bates (1996) adopt particular forms of that price of risk.

distributions by Levy (1985). Of these the portfolio comparisons and the market equilibrium (but not the SSD comparisons of terminal wealth) approaches have been extended to multiperiod problems by Perrakis (1986, 1988) and Ritchken and Kuo (1988).[11] Here we adopt the formulation of this latter study. We first reproduce the original results and then reformulate them in a stochastic dominance context.

Consider a two period economy with a stock with price S_0 and a bond with price B_0 at time 0. There is also a call option expiring at the end of the period of length T, and there are also n states of nature at T. The n states are ordered by the stock payoff s_T at option expiration. Thus, the stock pays s_j dollars in state j, where j is an index, such that $s_1 \leq s_2 \dots \leq s_n$ .The probabilities of the n states are $p_1, p_2, \dots, p_n$. The pricing kernel, the state-contingent discount factors, are denoted by $Y_1, \dots, Y_n$, and it is assumed that $Y_1 \geq Y_2 \geq \dots \geq Y_n$. The bond pays one dollar in each state. The call option has strike price K and payoffs $c_T = c_1, c_2, \dots, c_n$ in the n states of the economy. Then as shown in Ritchken (1985) the lower (upper) bounds of the option can be obtained by solving the following linear programming problems:

$$
\begin{aligned}
&\min(\max)\, c_0 = \sum_{j=1}^{n} c_j p_j Y_j \\
&\text{subject to:} \\
&S_0 = \sum_{j=1}^{n} s_j p_j Y_j \\
&B_0 = \sum_{j=1}^{n} p_j Y_j \\
&0 \leq p_j \leq 1 \\
&Y_1 \geq Y_2 \geq \dots \geq Y_n
\end{aligned}
\tag{2.1}
$$

The following assumptions constitute a set of *sufficient* conditions for the solution of (2.1) to yield stochastic dominance bounds for the option:

a. There exists at least one utility-maximizing risk averse investor in the economy who holds only the stock and the riskless asset;
b. There is no other trading date from time zero till the expiration of the option;[12]
c. The investor is marginal in the option market;

[11] Only the portfolio comparisons approach has been extended to cover transactions costs. See Constantinides and Perrakis (2002, 2007).

[12] Grundy (1991) presents an example of a two-period non-recombining binomial tree in which the call payoffs are not monotone non-decreasing with respect to the stock returns; a similar case is also in Cox and Rubinstein (1985, p. 157 footnote 14). It is important to note that the stochastic dominance approach is still valid in its recursive form in those cases, since condition (b) is violated in these examples; see also note 15.

d. The expected stock return $\hat{z}_n \equiv (\sum_{j=1}^{n} s_j p_j) / S_0$ is greater than or equal to the riskless return $R = B_0^{-1}$

Under assumptions (a)-(b) the pricing kernel is the normalized marginal utility of the investor wealth under the corresponding state of the economy, and the first two constraints in (2.1) reflect the first order conditions of utility maximization.[13] By assumption (d) the investor is always long in the stock, and risk aversion guarantees the monotonicity of the pricing kernel, the last constraint of (2.1). The investor may improve her utility by introducing a long or a short option position in her portfolio. Assumption (c) maintains the monotonicity of the pricing kernel in the presence of such an option position. The violation of either one of the two bounds defined by (2.1) triggers strategies that are utility-improving: an option price violating the upper (lower) of the bounds defined by (2.1) triggers a strategy of writing (purchasing) the violating option, which increases expected utility for any risk averse investor.

Assumption (a) is clearly the key condition underlying the stochastic dominance approach; it is, however, only a sufficient condition.[14] Assumption (b) will be relaxed in the next section. Assumption (c) can be satisfied if the option position of the investor is limited with respect to the size of her portfolio. Last, assumption (d) can easily be relaxed: the ordering of the state contingent discount factors shown in (2.1) is exactly reversed but the analysis still holds if $\hat{z}_n \leq R$; this case will be examined as an extension.

Assumption (a) may be restrictive for options on individual stocks, but its validity in the case of index options cannot be doubted, given that fact that numerous surveys have shown that a large number of US investors follow indexing strategies in their investments.[15] Note also that all option pricing models that combine arbitrage with equilibrium-type arguments use a monotone pricing kernel to price the options, which is the only feature arising out of assumption (a) used in deriving the bounds. A key factor in the stochastic dominance approach is the convexity of the option payoffs. Given such convexity, it can be shown that the two problems shown in (2.1) have the following solutions, which can be found with a simple geometric argument presented in Ritchken (1985)

$$\begin{aligned}
\overline{C}_0 &= \frac{1}{R}\left[\frac{R-\hat{z}_1}{\hat{z}_n-\hat{z}_1}\hat{c}_n + \frac{\hat{z}_n - R}{\hat{z}_n-\hat{z}_1}c_1\right] \\
\underline{C}_0 &= \frac{1}{R}\left[\frac{R-\hat{z}_h}{\hat{z}_{h+1}-\hat{z}_h}\hat{c}_{h+1} + \frac{\hat{z}_{h+1} - R}{\hat{z}_{h+1}-\hat{z}_h}\hat{c}_h\right]
\end{aligned} \tag{2.2}$$

---

[13] Under proportional transaction costs on the risky asset the investor utility is no longer a function of wealth but must contain the stock and riskless asset positions as separate arguments. The market equilibrium conditions are, therefore, considerably more complex. See Constantinides *et al* (2007).

[14] The monotonicity of the pricing kernel appears also as a necessary condition in several asset pricing models in which assumption (a) clearly does not hold. See the review monograph by Jackwerth (2004).

[15] Bogle (2005) reports that in 2004 index funds accounted for about one third of equity fund cash inflows since 2000 and represented about one seventh of equity fund assets.

In (2.2) we denote by $z_i, i = 1, \ldots, n$, the stock returns s_i / S_0, and we define the following conditional expectations for $j = 1, \ldots, n$:

$$\begin{aligned} \hat{c}_j &= \frac{\sum_{i=1}^{j} c_i p_i}{\sum_{i=1}^{j} p_i} = E\left[c_T \mid S_T \leq s_j \right] \\ \hat{z}_j &= \frac{\sum_{i=1}^{j} z_i p_i}{\sum_{i=1}^{j} p_i} = E\left[z_T \mid z_T \leq z_j \right] \end{aligned} \tag{2.3}$$

In the expressions (2.2) h is a state index such that $\hat{z}_h \leq R < \hat{z}_{h+1}$. The convexity of the option payoffs implies that the expressions $\hat{c}_j$ form a convex function of the conditional mean of the asset return $\hat{z}_j = E[S_T / S_0 \mid S_T / S_0 < z_j]$. The upper bound is a linear combination of the lowest return and the mean return, while the lower bound is a linear combination of the conditional expected returns $\hat{z}_h$ and $\hat{z}_{h+1}$. Therefore, the upper bound is a function of all the next period returns, while the lower bound is a function of $z_1, \ldots, z_{h+1}$.[16]

From the expressions (2.2) it can be easily seen that the two call option bounds are discounted expectations of the option payoff under two risk neutral probability distributions, as in the conventional binomial model. By substituting the conditional expected returns *j* in the bounds formulas, the upper bound can be expressed as the expectation of the payoff under the following risk neutral probability measure, the *U*-distribution

$$\begin{aligned} U_1 &= \frac{R - \hat{z}_1}{\hat{z}_n - \hat{z}_1} p_1 + \frac{\hat{z}_n - R}{\hat{z}_n - \hat{z}_1} \\ U_j &= \frac{R - \hat{z}_1}{\hat{z}_n - \hat{z}_1} p_j, \; j = 2, \ldots, n \end{aligned} \tag{2.4}$$

The lower bound, on the other hand, is the expectation of the option payoff in states $1, \ldots, h+1$ under the *L*-distribution, the risk neutral probability measure

$$\begin{aligned} L_j &= \frac{\hat{z}_{h+1} - R}{\hat{z}_{h+1} - \hat{z}_h} \frac{p_j}{\sum_{k=1}^{h} p_k} + \frac{R - \hat{z}_h}{\hat{z}_{h+1} - \hat{z}_h} \frac{p_j}{\sum_{k=1}^{h+1} p_k}, \; j = 1, \ldots, h \\ L_{h+1} &= \frac{R - \hat{z}_h}{\hat{z}_{h+1} - \hat{z}_h} \frac{p_{h+1}}{\sum_{k=1}^{h+1} p_k} \end{aligned} \tag{2.5}$$

Under the two probability measures the option bounds become[17]

---

[16] Note that in the binomial model (n = 2) the two bounds distributions coincide and define a unique option price; see Perrakis (1986). The stochastic dominance approach is, thus, a generalization of the binomial model.

[17] Hereafter a superscript denotes the measure under which the expectation is taken.

$$\overline{C}(S) = \frac{1}{R} E^{U}[\max(Sz - K, 0)]$$
$$\underline{C}(S) = \frac{1}{R} E^{L}[\max(Sz - K, 0)] \tag{2.6}$$

where $\overline{C}(S)$ and $\underline{C}(S)$ are convex functions.

From these expressions it can be seen that the upper bound distribution is a mixture of the historical or subjective probability distribution $P(z)$ (the P-distribution) of the underlying asset return $z = S_T / S_0$, and a distribution concentrated at $z_{\min}$, the lowest possible asset return.

$$U(z) = \begin{cases} P(z) & \text{with probability } \frac{R - z_{\min}}{E(z) - z_{\min}} \\ 1_{z_{\min}} & \text{with probability } \frac{E(z) - R}{E(z) - z_{\min}} \end{cases} \tag{2.7}$$

This general formula applies both for continuous and discrete distributions of the returns. The weight of the lowest value in this mixture is such that *U(z)* is a risk neutral distribution. The transformation shown in (2.7) does not hold for distributions *P(z)* with $z_1 = z_{\min} = \min\{z\} = 0$. In this case, the upper bound becomes

$$\overline{C} = \frac{1}{E(z)} E^{P}[\max(Sz - K, 0)] \tag{2.8}$$

The upper bound in this case is obtained by discounting the expected option payoff at the expected rate of return of the stock price.

Similarly, the lower bound distribution is a mixture of two truncated multinomial distributions

$$\overline{C} = \frac{1}{E(z)} E^{P}[\max(Sz - K, 0)] \tag{2.9}$$

$$L(z) = \begin{cases} P(z \mid \hat{z} < \hat{z}_h) & \text{with probability } \frac{\hat{z}_{h+1} - R}{\hat{z}_{h+1} - \hat{z}_h} \\ P(z \mid \hat{z} < \hat{z}_{h+1}) & \text{with probability } \frac{R - \hat{z}_h}{\hat{z}_{h+1} - \hat{z}_h} \end{cases} \tag{2.10}$$

If the underlying asset has a continuous distribution, the probability for computing the lower bound is the truncated distribution

$$L(z) = P(z \mid E(z) \le R) \tag{2.11}$$

This distribution is obtained by truncating the given returns distribution till the expected return becomes equal to the riskless rate.

An alternative representation of the bounds that links the stochastic dominance approach to the conventional equilibrium valuation of contingent claims is by taking the expectation with the measure $P(z)$ of the call payoff multiplied by a stochastic discount factor Y^U *or* Y^L for the upper and lower bounds respectively. In such a case we have

$$\begin{aligned} \overline{C}(S) &= E^P[Y^U \max(Sz-K,0)] \\ \underline{C}(S) &= E^P[Y^L \max(Sz-K,0)] \end{aligned} \tag{2.12}$$

with the factors Y^U and Y^L given by the following expressions for continuous distributions $P(z)$

$$\begin{aligned} Y^U(z) &= \frac{1}{R}(1-q) + \frac{1}{R} q\delta(z_{\min}) \\ Y^L(z) &= \begin{cases} \frac{1}{R}\frac{1}{P(z \le z_h)} & z \le z_h \\ 0 & z > z_h \end{cases} \end{aligned} \tag{2.13}$$

where $E(z \mid z \le z_h) = R$, $\delta(\cdot)$ is Dirac's delta and $q = \dfrac{E(z)-R}{E(z)-z_{\min}}$. This representation will also be useful in the multiperiod formulation of the next section.

Last, we present the single period option bounds for the case where $\hat{z}_n < R$. Although this case is not very relevant in either single period or multiperiod iid returns, it may arise in individual subperiods for non-iid returns when the parameters of the return distribution are functions of the stock price at the beginning of the subperiod. It may also arise in the pricing of options on the exchange rate. The expressions for $\overline{C}(S)$ and $\underline{C}(S)$ are very similar to (2.2). Instead of (2.3) we now define the conditional expectations

$$\bar{c}_j = \frac{\sum_{i=j}^{n} c_i p_i}{\sum_{i=j}^{n} p_i} = E\left[c_T \mid s_T \ge s_j\right] \quad \bar{z}_j = \frac{\sum_{i=j}^{n} z_i p_i}{\sum_{i=j}^{n} p_i} = E\left[z_T \mid z_T \ge z_j\right] \tag{2.3'}$$

Then instead of (2.2) we get

$$\begin{aligned} \underline{C}_0 &= \frac{1}{R}\left[\frac{\bar{z}_{h+1}-R}{\bar{z}_{h+1}-\bar{z}_h}\bar{c}_h + \frac{R-\bar{z}_h}{\bar{z}_{h+1}-\bar{z}_h}\bar{c}_{h+1}\right] \\ \overline{C}_0 &= \frac{1}{R}\left[\frac{\bar{z}_n-R}{\bar{z}_n-\bar{z}_1}\bar{c}_1 + \frac{R-\bar{z}_1}{\bar{z}_n-\bar{z}_1}\bar{c}_n\right] \end{aligned} \tag{2.2'}$$

Here again the two sates $\bar{z}_h$ and $\bar{z}_{h+1}$ are defined from the relation $\bar{z}_h \le R \le \bar{z}_{h+1}$. For a continuous return distribution $P(z)$ the risk neutral distributions $U(z)$ and $L(z)$ of the upper and lower bounds respectively become, instead of (2.7) and (2.10)

$$U(z) = \begin{cases} P(z) & \text{with probability } \dfrac{z_{\max} - R}{z_{\max} - E(z)} \\ 1_{z_{\max}} & \text{with probability } \dfrac{R - E(z)}{z_{\max} - E(z)} \end{cases}, \tag{2.7'}$$

$$L(z) = P(z \mid E(z) \geq R). \tag{2.10'}$$

We close this section by discussing briefly the case where the option payoff is not convex. In such a case the expressions $\hat{c}_j$ are no longer a convex function of the conditional mean of the asset return $\hat{z}_j = E[S_T / S_0 \mid S_T / S_0 < z_j]$ in (2.3). In fact, the $\hat{c}_j$'s may not even form an increasing function of the conditional means. Nonetheless, the program (2.1) yields a maximum and a minimum that are found on the *convex hull* of the graph of the $\hat{c}_j$'s plotted as functions of the conditional means. Although closed form solutions for the bounds don't exist in this case, their values may be computed numerically for one period, and then extended recursively in the multiperiod case as in the next section. These cases are particularly important when the returns are not iid, as in the case of stochastic volatility examined in section 6.

3. Multiperiod Bounds: the General Case

It is easy to extend the bounds to a multiperiod context by adopting assumptions (a), (c) and (d) of the previous section. Assume that there exists at least one investor in the market holding only the underlying asset, the riskless asset and possibly the option. The investor maximizes recursively the expectation of a concave utility function $u(W_{T'})$ over time periods $0,1,...,T'-1$ of the terminal wealth $W_{T'}$ at time *T'*, a horizon that is equal to or larger than the time *T* to option expiration. Alternatively, the investor objective may be to maximize the discounted stream of the utilities of consumption at every time period, plus possibly a concave function of a final endowment. In either case it is well known that in the absence of transaction costs the value function, the indirect utility at any intermediate time point, is a concave function of wealth at that point. Given assumptions (c) and (d) that guarantee that the investor would always be long in the stock, the concavity of the indirect utility is sufficient to establish the monotonicity of the state contingent discount factors in every period. The derivation of the multiperiod option bounds is simplest in the case of iid returns, which are examined first.

A. The bounds for iid returns

Under iid returns the single period distribution $P(z)$ (discrete or continuous) of the one-period stock return $z = S_{t+1} / S_t$ remains the same in each time period. Similarly, the risk neutral distributions $U(z)$ and $L(z)$ remain the same in every period, given by (2.4) and (2.5) or (2.7) and (2.10) in the discrete and continuous cases, respectively. It is then very simple to prove by induction that the multiperiod option bounds can be derived by a recursive version of the linear programming problem (2.1). At any time $t < T$ the pricing kernel $Y_{t+1,1}, \ldots, Y_{t+1,n}$ is monotone because of the concavity of the indirect utility and assumptions (c) and (d), that guarantee that the investor will always be long in the stock. At each iteration the convex functions $\overline{C}(Sz_j)$ and $\underline{C}(Sz_j)$ will replace the call payoffs c_j in the maximization and minimization problems respectively. Given a current stock price S_t at time $t \le T$, and with the superscripts U and L denoting expectations under the corresponding distributions $U(z)$ and $L(z)$, the option bounds $\overline{C_t}(S_t)$ and $\underline{C}_t(S_t)$ become[18]

$$
\begin{aligned}
\overline{C_t}(S_t) &= \frac{1}{R} E^U [\overline{C}_{t+1}(S_t z) | S_t] \\
\underline{C}_t(S_t) &= \frac{1}{R} E^L [\underline{C}_{t+1}(S_t z) | S_t] \\
\overline{C_T}(S_T) &= \underline{C}_T(S_T) = \max\{S_{T-1} z - K, 0\}
\end{aligned}
\tag{3.1}
$$

In the special case where $z_1 = \min\{z\} = 0$ and (2.8) holds in a single period the upper bound $C_t(S_t)$ can be estimated very simply by the law of iterated expectations. It is given by[19]

$$
\overline{C_t}(S_t) = \frac{1}{[E(z)]^{T-t}} E^P [\max(S_T - K, 0) | S_t] \qquad (3.1)'
$$

In a distribution with iid returns the multiperiod bounds can be extended to American options with very little reformulation. The following expressions give the bounds for American call options on stocks with a constant dividend yield γ, as well as for American put options. They may be proven very simply by induction.

$$
\begin{aligned}
\overline{C}_{A,t}(S_t) &= \frac{1}{R} E^U [\max\{S_t z_{t+1}(1+\gamma) - K, \overline{C}_{A,t+1}(S_t z_{t+1})\} | S_t], \\
\underline{C}_{A,t}(S_t) &= \frac{1}{R} E^L [\max\{S_t z_{t+1}(1+\gamma) - K, \underline{C}_{A,t+1}(S_t z_{t+1})\} | S_t], \\
\overline{C}_{AT}(S_T) &= \underline{C}_{AT}(S_T) = (S_{T-1} z_T (1+\gamma) - K)^+.
\end{aligned}
\tag{3.2}
$$

---

[18]For a more detailed analysis see Perrakis (1986, 1988) and Ritchken and Kuo (1988).

[19]Note that (3.1)' also holds in the presence of proportional transaction costs in the underlying asset if it is multiplied by the roundtrip transaction costs. See Proposition 1 of Constantinides and Perrakis (2002).

$$\overline{P}_{A,t}(S_t) = \frac{1}{R} E^U[\max\{K - S_t z_{t+1}, \overline{P}_{A,t+1}(S_t z_{t+1})\} | S_t],$$
$$\underline{P}_{A,t}(S_t) = \frac{1}{R} E^L[\max\{K - S_t z_{t+1}, \underline{P}_{A,t+1}(S_t z_{t+1})\} | S_t]. \quad (3.3)$$
$$\overline{P}_{A,T}(S_T) = \underline{P}_{A,T}(S_T) = (K - S_{T-1} z_T)^+$$

B. The general case of non-iid returns

In a general model of security returns in which the iid assumption does not hold the option price at time t under the no arbitrage approach is the conditional expectation of its payoff times the stochastic discount factor given the information available at time t. The same result applies also in the stochastic dominance approach under assumptions (a) and (c) of the previous section, except that that the discount factor now has the interpretation of the normalized marginal utility of wealth. Let $(\Omega, \mathcal{F}, P)$ a complete probability space where Ω comprises all the possible sequences of states of the economy until the expiration of the option, $\mathcal{F}$ is the sigma algebra generated by Ω and P is the subjective probability measure representing the beliefs of an investor. The option price at time t is

$$c_t = E^P[\max(S_T - K, 0) Y_T \mid \mathcal{F}_t]$$

Where $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_T$ is the filtration generated by Ω, $\mathcal{F}_t$ represents the information set available to the investor at time *t*, and Y_T is the pricing kernel. By the law of iterated expectations

$$\begin{aligned} c_t &= E^P[E^P[\max(S_T - K, 0) Y_T \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t] \\ &= E^P[c_{t+1} Y_{t+1} \mid \mathcal{F}_t] \end{aligned}$$

Two problems arise when the iid assumption is relaxed. The first one is the possibility of non-convexities of the call option bounds with respect to the price of the underlying asset at any time $t < T$. In a single period the convexity of the bounds is strictly a function of the convexity of the call payoff with respect to the price of the underlying asset. In a multiperiod or in a continuous time model the convexity of the payoff is not always sufficient to guarantee the convexity of the option price. This issue has been examined in detail by Bergman *et al* (1996), who provide necessary and sufficient conditions for convexity to hold. They show that convexity holds in all univariate diffusion cases, as well as in most diffusion cases in which volatility is stochastic.

The second problem that arises in applying the stochastic dominance approach in non-iid returns is the fact that assumption (d) may no longer hold in every period, since it may turn out that in the conditional distribution given $\mathcal{F}_t$ we may have $\hat{z}_n < R$. For all cases in which assumption (d) holds problem (2.1) can be rewritten in recursive form as follows

$$
\begin{aligned}
&\min(\max)\ c_t = \sum_{j=1}^{n} c_{t+1,j}\, p_{t+1,j}\, Y_{t+1,j} \\
&\quad \text{subject to:} \\
&\qquad S_t = \sum_{j=1}^{n} s_{t+1,j}\, p_{t+1,j}\, Y_{t+1,j} \\
&\qquad B_t = \sum_{j=1}^{n} p_{t+1,j}\, Y_{t+1,j} \\
&\qquad 0 \le p_{t+1,j} \le 1 \\
&\qquad Y_{t+1,1} \ge Y_{t+1,2} \ge \ldots \ge Y_{t+1,n}
\end{aligned}
\tag{3.4}
$$

where the variables $s_{t+1}, c_{t+1}, p_{t+1}, Y_{t+1}$ are conditional on the information available at time *t*. If, on the other hand, $\hat{z}_n < R$ then the monotone ordering of the state contingent discount factors is reversed in (3.4). In both cases if, in addition, convexity also holds then, by applying successively the results of the two-period problem, we can find expressions equivalent to (3.1) for non-iid returns. For instance, the upper bound yields at every time period

$$c_t \le \frac{1}{R} E^U[c_{t+1} \mid \mathcal{F}_t] \le \frac{1}{R} E^U[\overline{C}_{t+1} \mid \mathcal{F}_t]$$

and by repeatedly applying the law of iterated expectations we get

$$\overline{C}_t = \frac{1}{R^n} E^{U^{(n)}}[\max(Sz^{(n)} - K, 0) \mid \mathcal{F}_t]$$

where $z^{(n)}$ is the n period return and $U^{(n)}$ is its distribution. A similar process may also be applied for the derivation of the lower bound. When the returns are iid $U^{(n)}$ is the the *n*-period convolution of the upper bound distribution derived in the previous section. In the more general case the distribution of S_{t+1} may depend on S_t but the option price may continue to be convex in S_t. In such cases the option upper bound is still given recursively by[20] (3.1), with the distribution U given by (2.7) or (2.7)' depending on whether $\hat{z}_n \ge R$ or $\hat{z}_n < R$, and with the return distribution $P(z)$ now dependent on S_t. While a closed form solution for the bound would be difficult to derive, the bounds may be easily computed via Monte Carlo simulations of the distributions $U(z)$ and $L(z)$ generated at each time step from the corresponding $P(z)$. For both bounds the monotone ordering of the state contingent discount factors is reversed in (3.4) when $\hat{z}_n < R$, and it is (2.2)' and (2.3)', or (2.7)' and (2.10)', that provide the appropriate bounds for that particular subperiod.

[20]A similar recursion can be applied when $z_1 = \min\{z\} = 0$ and (2.8) applies in every period.

In the remaining sections of this paper we examine the multiperiod bounds given by (3.1) or (3.4) in several cases of practical interest. It is the limiting behavior of these expressions as trading becomes progressively more dense and we pass from discrete to continuous time that is of interest in this paper. It will be shown that this limiting behavior contains as special cases virtually the entire set of option pricing models that have appeared in the literature till now, most of them derived by arbitrage methods.

4. Option Bounds in Continuous Time: the Diffusion Case

The limiting behavior of the bounds in (3.1) was examined in Perrakis (1988) for the special case of a stock return following a trinomial distribution. It was shown that when that distribution tended to a diffusion process the limit of both upper and lower bounds under a monotone pricing kernel was the Black-Scholes option price. The convergence criteria used in that study were the ones provided by Merton (1982) for iid returns following a general multinomial process. Since the bounds are available in closed form in such a case, it suffices to show that the limiting form of the multiperiod convolutions of the distributions *U(z)* and *L(z)* given by (2.7) and (2.9) or (2.10) is a risk neutral diffusion with the same constant volatility as the initial process.

This line of approach is, unfortunately, not available when the underlying stock returns are not iid. Although the Merton (1982) criteria for the convergence to a diffusion of the multinomial discretization of the underlying stochastic process are still valid, they are not very useful in characterizing the limiting process. Further, the option bounds themselves are available only as recursive expressions of time-varying distributions, whose limiting form is not easy to ascertain under general conditions. For this reason we shall examine the behavior of the bounds for a more general diffusion process by adopting a different discretization of the stochastic process and a more general approach to convergence analysis.

We consider the most general case of a diffusion process followed by the stock price in continuous time

$$dS = \mu(S,t)dt + \sigma(S,t)dW, \tag{4.1}$$

where W is a Wiener process with $E(dW)=0$ and $Var(dW)=dt$ and $\mu(S,t),\sigma(S,t)$ are unspecified functions; in the traditional Black-Scholes model both functions are linear with constant coefficients, $\mu(S,t)=\mu S$ and $\sigma(S,t)=\sigma S$. We seek a discrete time Markovian stochastic process over the interval $[0,T]$ to option expiration that would converge to (4.1) as the length Δt of the elementary time period tends to zero. Such a discrete time process defines for any number m of time periods to expiration a sequence of stock prices $\{S_t \mid \Delta t, m\}$ and an associated probability measure P^m. The weak convergence property for such processes[21] stipulates that for any continuous bounded

[21]For more on weak convergence for Markov processes see Ethier and Kurz (1986), or Strook and Varadhan (1979).

function f we must have $E^{P_m}\left[f(S_T^m)\right] \to E^P\left[f(S_T)\right]$, where the measure P corresponds to the process given by (4.1). P_m is then said to converge weakly to P and S_T^m is said to converge in distribution to S_T. In our case, once weak convergence to (4.1) has been established for the chosen discretization we must examine the convergence of the transformations U and L of P corresponding to the upper and lower call option bounds respectively. The call option price bounds would then be the limits of the expectation of the call payoff under the limiting distributions, and a unique option price results if both U and L converge to the same limit.

There are several ways to verify the weak convergence of Markov processes. For instance, a necessary and sufficient condition for the convergence to a diffusion is the Lindeberg condition, which was used by Merton (1982) to develop criteria for the convergence of multinomial processes. Let x_t denote a discrete stochastic process in *d*-dimensional space. The Lindeberg condition, a necessary and sufficient condition that x_t converges weakly to a diffusion, is that for any fixed $\delta > 0$ we must have

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\|y-x\|>\delta} Q_{\Delta t}(x, dy) = 0 \tag{4.2}$$

where $Q_{\Delta t}(x, dy)$ is the transition probability from $x_t = x$ to $x_{t+\Delta t} = y$ during the time interval Δt. Intuitively, it requires that x_t does not change very much when the time interval Δt goes to zero.

When the Lindeberg condition is satisfied, the following limits exist

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\|y-x\|<\delta} (y_i - x_i) Q_{\Delta t}(x, dy) = \mu_i(x) \tag{4.3}$$

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\|y-x\|<\delta} (y_i - x_i)(y_j - x_j) Q_{\Delta t}(x, dy) = \sigma_{ij}(x) \tag{4.4}$$

The conditions (4.2), (4.3) and (4.4) are equivalent to the weak convergence of the discrete process to a diffusion process with the generator

$$\mathcal{A} = \frac{1}{2} \sum_{i=1,\, j=1}^{d} \sigma_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \mu_i \frac{\partial}{\partial x_i} . \tag{4.5}$$

By the definition of the generator, for each bounded, real valued function f we have

$$\lim_{\Delta t \to 0} \frac{Ef(x_{t+\Delta t}) - f(x_t)}{\Delta t} = \mathcal{A} f \qquad (4.6)$$

The limit diffusion process can also be described by the d-dimensional stochastic differential equation

$$dx = \mu(x,t)dt + \sigma(x,t)dW$$

which corresponds in our case to the uni-dimensional equation (4.1) when $x = S$. To make the convergence results meaningful, it is assumed that this differential equation has a solution.[22] We shall examine the convergence of the option bounds in continuous time under the following discrete time representation of the stock return process

$$S_{t+\Delta t} - S_t = \mu(x_t,t)\Delta t + \sigma(x_t,t)\varepsilon_{t+\Delta t}\sqrt{\Delta t} \qquad (4.7)$$

In (4.7) x_t is a vector of state variables known at time t and including, but not necessarily limited to S_t, and $\varepsilon_{t+\Delta t}$ has a bounded continuous distribution of mean zero and variance one, $\varepsilon_{t+\Delta t} \sim D(0,1)$ and $0 < \varepsilon_{\min} \le \varepsilon_{t+\Delta t} \le \varepsilon_{\max}$, but otherwise unrestricted. First we establish that this representation is a valid discretization of the continuous time stochastic process by means of the following result, whose proof is in the appendix.

Lemma 1. *The discrete process described by equation (4.7) converges weakly to the diffusion (4.1).*

Once we have proven the lemma we then apply it to the process governing the stock return[23]

$$(S_{t+\Delta t} - S_t)/S_t \equiv z_{t,t+\Delta t} = \mu(x_t,t)\Delta t + \sigma(x_t,t)\varepsilon_{t+\Delta t}\sqrt{\Delta t}. \qquad (4.8)$$

The limit of (4.8) is the following continuous time process

$$\frac{dS}{S} = \mu(x_t,t)dt + \sigma(x_t,t)dW \qquad (4.9)$$

where $\mu(\cdot)$ and $\sigma(\cdot)$ are functions of x_t such that equation (4.9) has a solution. Suppose now that we have chosen a certain Δt and are at some time $t < T$, observing the state variables x_t in (4.8) and ascertaining that $\mu > R - 1 \equiv r$. We have already evaluated the option bounds $\overline{C}(S_{t+\Delta t}) = \overline{C}(S_t(1+z_{t,t+\Delta t}))$ and $\underline{C}(S_{t+\Delta t}) = \underline{C}(S_t(1+z_{t,t+\Delta t}))$. We then

---

[22] For instance, a Lipschitz condition for the functions $\mu(\cdot)$ and $\sigma(\cdot)$ provides the existence and uniqueness of a strong solution. This condition is satisfied by all the models used in option pricing.

[23]Note that $z_{t,t+\Delta t}$ differs from the variable z used in equation (3.1) since the latter is the wealth relative while the former is the wealth relative minus one.

evaluate the upper and lower bounds at t by applying the relations[24]

$$\begin{aligned}\overline{C}(S_t) &= E^U[C(S_t(1+z_{t,t+\Delta t}))\mid \mathcal{F}_t]\\ \underline{C}(S_t) &= E^L[C(S_t(1+z_{t,t+\Delta t}))\mid \mathcal{F}_t]\end{aligned} \tag{4.10}$$

where $\overline{C}(S_T) = \underline{C}(S_T) = \max\{S_T - K, 0\}$, the superscripts U and L denote the transformations of the distribution of the one-period return given in (2.7) and (2.10) respectively, with the return $z_{t,t+\Delta t}$ given by (4.8). If, on the other hand, the state variables x_t are such that $\mu \le r$ then the expectations are taken with respect to the transformations of the distribution given by (2.7)' and (2.10)'.

With such a procedure we can then generate the upper and lower bounds at time zero corresponding to any path of stock prices drawn from the discrete process (4.8); the distribution of the error term $\varepsilon_{t,t+\Delta t}$ can be a very simple one, such as a uniform distribution centered at 0 with its variance set equal to 1. Of particular interest, however, is the existence of a limit to these bounds as $\Delta t \to 0$ and (4.8) tends to (4.9). These limits are expressed by the following two propositions that form the main results of this section and whose proof is in the appendix.

Proposition 1. *When the underlying asset follows a continuous time process described by (4.9) the stochastic dominance upper bound of a European call or put option converges to the discounted expectation of the terminal payoff of an option on an asset whose dynamics are described by the process*

$$\frac{dS}{S} = r\,dt + \sigma(x_t, t)\,dW \tag{4.11}$$

where r *is the (continuous time) riskless rate of interest.*

Proposition 2. *Under the conditions of Proposition 1 the stochastic dominance lower bound of a European call or put option converges to the same limit as the upper bound.*

These remarkable results establish the formal equivalence of the stochastic dominance approach to the prevailing arbitrage methodology for plain vanilla option prices whenever the underlying asset dynamics are generated by a diffusion or Ito process, no matter how complex. The equivalence holds for stock options as well as for index options, even though the assumption that there must be an investor holding only the stock and the riskless asset may not always appear reasonable. Note that the univariate Ito process is the only type of asset dynamics, corresponding to dynamically complete markets, for which options can be priced by arbitrage considerations alone. The

[24] The recursive evaluation shown in (4.10) assumes that the recursive evaluation of the bounds shown in (3.4) is still valid, which is the case only if the option price is convex at any time t with respect to the price of the underlying asset. While this holds always for the univariate diffusions, the cases of multivariate diffusions like stochastic volatility are more complex and will be examined in Section 6. If convexity does not hold (4.10) still holds but the recursive expectations do not necessarily bind all admissible option prices.

stochastic volatility, GARCH processes, and mixed jump-diffusion models need additional assumptions beyond arbitrage in order to complete the market.

Valuation of the option by Monte Carlo simulation of the bounds for the cases where no closed-form expression for the option price exists is certainly an alternative to an option value computed as a discounted payoff of paths generated by the Monte Carlo simulation of (4.11). While there may not be any computational advantages in going through the bounds route to option valuation, the fact that both upper and lower bound tend to the same limit may provide a benchmark for the accuracy of the valuation, in contrast to the direct simulation of (4.11). Further, the possibility that the bounds may provide an accurate approximation of the option price in cases where the exact form of the asset dynamics is unknown is an empirical question that merits consideration, even though it lies beyond the scope of this paper. In the following sections we examine the behavior of the bounds under asset dynamics that go beyond the general process for dynamically complete markets described by (4.1).

5. Option Bounds in Continuous Time and Incomplete Markets I: Mixed Jump-Diffusion Processes

Jump-diffusion processes characterize the dynamics of the underlying asset price distribution whenever there are discontinuous jumps in the time path of the stock price caused by the sudden and unexpected arrival of important information. Such jumps have long been recognized as an important source of market incompleteness. Their presence makes the valuation of options solely by arbitrage methods infeasible, except in a binomial model.[25] As for the stochastic dominance approach, it was shown that the two bounds converge to two different option values at the limit of continuous trading even in the case of a very simple three-state jump process (up, down, and stay the same).[26]

When there are jumps in the underlying asset price distribution it is not possible to replicate the option with a portfolio comprising the riskless asset and the underlying asset. The pricing of the option requires extra assumptions regarding the jump risk. The most common assumption, originally introduced by Merton (1976), is that the jump risk is diversifiable. In such a case the market will not pay a risk premium over the riskless rate for bearing the jump risk and risk neutral pricing applies by assuming that the jump probabilities are risk neutral. With such an assumption a closed-form expression for the option price was provided by Merton for the case where the amplitude of the jump size follows a lognormal distribution. Alternative approaches for valuing options in jump-diffusion cases have been provided by Amin and Ng(1993), Amin (1993) and Bates (1991, 1996).[27]

---

[25]See Cox and Rubinstein (1985, pp. 365-368).

[26]See Proposition 6 in Perrakis (1988).

[27]See also Bakshi, Cao and Chen (1997), who added jump components to a stochastic volatility model. More recently Duffie, Pan and Singleton (2000) have introduced option pricing models for underlying assets that contain jumps in both asset returns and their stochastic volatility.

In this section we examine the stochastic dominance approach to option pricing in the case of underlying assets whose returns follow jump-diffusion processes. As with the general diffusion case, we first provide a discretization of the continuous time process that converges at the limit to the given jump-diffusion process. The option bounds are derived by the stochastic dominance approach from such a discretization by applying the risk neutral transformations (2.7) and (2.10) to the discrete one-period distribution. The two transformed distributions are then shown to converge at the continuous time limit to two different option prices. These prices correspond to two different risk neutral jump-diffusion processes, each one of which prices options in a manner similar to the Merton (1976) assumption of diversifiable jump risk. We provide two partial differential equations (pde) satisfied by the upper and lower bounds respectively.[28] The two bounds contain all the jump-diffusion option prices that have appeared in earlier studies, including the Merton (1976) price. We assume that the underlying asset returns follow the process

$$\frac{dS_t}{S_t} = (\mu_t - \lambda\mu_J)dt + \sigma_t dW_t + J_t dN_t \tag{5.1}$$

where the last term is a jump component added to the diffusion. Although our results apply to the more general case where both μ_t and σ_t are functions of S_t and time, we shall assume in what follows that $\mu_t = \mu, \sigma_t = \sigma, \lambda_t = \lambda$ and $J_t = J$, in line with earlier studies; we shall also assume that $\mu > r$. The variable J represents the logarithm of the jump size. It is a random variable with distribution D_J with mean μ_J and variance σ_J. N is a Poisson counting process with intensity λ. In most of the literature it is assumed that D_J is a normal distribution.

The first step in deriving the bounds on this process is to find a discrete approximation that converges weakly to (5.1). It will be shown that the following process is such an approximation, with $z_{t,t+\Delta t}$ denoting the logarithmic return.

$$z_{t,t+\Delta t} = (\mu - \lambda\mu_J)\Delta t + \sigma\varepsilon\sqrt{\Delta t} + J\Delta N \tag{5.2}$$

where ε is a random variable with a given distribution, either a bounded continuous distribution $D(0,1)$, or even a simple binomial process taking the values ± 1 with probability 1/2. The returns process can be characterized as a mixture of a diffusion and a jump, with corresponding probabilities $1-\lambda\Delta t$ and $\lambda\Delta t$:

$$z_{t,t+\Delta t} = \begin{cases} z_D = (\mu - \lambda\mu_J)\Delta t + \sigma\varepsilon\sqrt{\Delta t} & \text{with probability} \quad 1-\lambda\Delta t \\ J & \text{with probability} \quad \lambda\Delta t \end{cases}$$

[28] Of these two pde's only the one corresponding to the upper bound yields a closed-form solution under a lognormal distribution of the amplitude of the jumps. For the lower bound, and for all other jump amplitude distributions, both option bounds can be obtained either through numerical methods or through their characteristic functions following the approach of Heston (1993) and Bates (1996).

It can be easily seen that this process does not satisfy the Lindeberg condition, since

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} Q_t(\delta) = \lambda \int_{|z_{t,t+\Delta t}|>\delta} dD_J(J) + \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|z_{t,t+\Delta t}|>\delta} (1-\lambda \Delta t) dD_D(\varepsilon)$$

As shown in the proof of Lemma 1 for the diffusion case, the second integrand is zero for Δt sufficiently low. However, the first integrand is strictly positive for any Δt, implying that the process does not converge to a diffusion in continuous time. The following result, proven in the appendix, shows that (5.2) is a valid discrete time representation of (5.1).

Lemma 2. *The discrete process described by (5.2) converges weakly to the jump-diffusion process (5.1).*

Next we examine the limiting behavior of the stochastic dominance bounds derived from (5.2). We assume, without loss of generality, that the variable *J* takes both positive and negative values, or that $J_{\min} < 0 < J_{\max}$, implying that the jump amplitude takes values both above and below 1. For the option upper bound we apply (2.7) to the discrete time process defined by (5.1). For such a process we note that as Δt decreases, there exists *h*, such that for any $\Delta t \le h$, the minimum outcome of the jump component is less than the minimum outcome of the diffusion component, $J_{\min} < \mu_t \Delta t + \sigma_t \varepsilon_{\min} \sqrt{\Delta t}$. Consequently, for any $\Delta t \le h$, the minimum outcome of the returns distribution is $J_{\min}$, which is the value that we substitute for $z_{\min}$ in (2.7). With such a substitution we have now the following result, proven in the appendix.

Proposition 3. *When the underlying asset follows a jump-diffusion process described by (5.1) the upper option bound is the discounted expected payoff of an option on an asset whose dynamics are described by the jump-diffusion process*

$$\frac{dS_t}{S_t} = \left[r - (\lambda + \lambda_U)\mu_J^U \right] dt + \sigma dW + J^U dN_t \tag{5.3}$$

where r is the riskless interest rate,

$$\lambda_U = -\frac{\mu - r}{J_{\min}} \tag{5.4}$$

and J^U is a jump with the distribution and mean

$$\begin{aligned} J^U &= \begin{cases} J & \text{with probability } \frac{\lambda}{\lambda+\lambda_U} \\ J_{\min} & \text{with probability } \frac{\lambda_U}{\lambda+\lambda_U} \end{cases} \\ \mu_J^U &= \frac{\lambda}{\lambda+\lambda_U}\mu_J + \frac{\lambda_U}{\lambda+\lambda_U} J_{\min} \end{aligned} \tag{5.5}$$

The proof of Proposition 3 comes from the convergence of the risk-neutral discrete time process defined by (5.2) to the jump-diffusion process given by (5.3). Given now Proposition 3, we can then use the results derived by Merton (1976) for options on assets following jump-diffusion processes with the jump risk fully diversifiable.[29] Applying Merton's approach to (5.3) we find that the upper bound on call option prices for the jump-diffusion process (5.1) must satisfy the following pde, with terminal condition $C(S_T, T) = \max\{S_T - K, 0\}$:

$$\left[r - (\lambda + \lambda_U)\mu_J^U\right] S \frac{\partial \overline{C}}{\partial S} - \frac{\partial \overline{C}}{\partial T} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \overline{C}}{\partial S^2} + \lambda_U E^U[\overline{C}(SJ^U) - \overline{C}(S)] - r\overline{C} = 0 \quad (5.6)$$

An important special case is when the lower limit of the jump amplitude is equal to 0, in which case $J_{\min} = -\infty$ and the process (2.7) is replaced by (2.8). In such a case *r* is replaced by μ in (5.6), which now becomes

$$[\mu - \lambda\mu_J] S \frac{\partial \overline{C}}{\partial S} - \frac{\partial \overline{C}}{\partial T} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \overline{C}}{\partial S^2} + \lambda E[\overline{C}(SJ) - \overline{C}(S)] - \mu\overline{C} = 0 \quad (5.7)$$

If (5.7) holds and we assume, in addition, that the amplitude of the jumps has a lognormal distribution with $J \sim N(\mu_J, \sigma_J)$, the distribution of the asset price given that *k* jumps occurred is conditionally normal, with mean and variance

$$\begin{aligned} \mu_k &= \mu - k\lambda\mu_J + \frac{k}{T}\ln(1 + \mu_J) \\ \sigma_k^2 &= \sigma^2 + \frac{k}{T}\sigma_J^2 \end{aligned} \quad (5.8)$$

Hence, if *k* jumps occurred, the option price would be a Black-Scholes expression with μ_k replacing the riskless rate *r*, or $BS(S, K, T, \mu_k, \sigma_k)$. Integrating (7.7) would then yield the following upper bound, which can be obtained directly from Merton (1976) by replacing *r* by μ.

$$\overline{C} = \sum_{k=0}^{\infty} \exp[-\lambda(1 + \mu_J)T] \frac{[\lambda(1 + \mu_J)T]^k}{k!} BS(S, K, T, \mu_k, \sigma_k) \quad (5.9)$$

When the jump distribution is not normal, the conditional asset distribution given k jumps is the convolution of a normal and k jump distributions. The upper bound cannot be obtained in closed form, but it is possible to obtain the characteristic function of the bound distribution. Similar approaches can be applied to the integration of equation (5.6), which holds whenever $0 > J_{\min} > -\infty$. Closed form solutions can also be found whenever the amplitude of the jumps is fixed as, for instance, when there is only an up and a down jump of a fixed size.[30] A pde similar to (5.6) also holds if the process has only "up" jumps, in which case $J_{\min} = 0$ and the lowest return $z_{\min}$ in (2.7) comes from the

[29] Remark that in the stochastic dominance approach, we do not assume that the jump risk is diversifiable.

[30] See, for instance, the example in Masson and Perrakis (2000).

diffusion component.

Next we examine the option lower bound for the jump-diffusion process given by (5.1) and its discretization (5.2). We apply now (2.10) to the process (5.2) and we prove in the appendix the following result.

Proposition 4. *When the underlying asset follows a jump-diffusion process described by (5.1) , the lower option bound is the discounted expected payoff of an option on an asset whose dynamics is described by the jump-diffusion process*

$$\frac{dS_t}{S_t} = \left[r - \lambda \mu_J^L \right] dt + \sigma_t dW_t + J^L dN_t \tag{5.10}$$

where J^L *is a jump with the truncated distribution*

$$J \mid J \leq \overline{J}$$

The mean of the jump and the value of $\overline{J}$ *can be obtained by solving the equations*

$$\begin{aligned} &\mu - \lambda \mu_J + \lambda \mu_J^L = r \\ &\mu_J^L = E(J \mid J \leq \overline{J}) \end{aligned} \tag{5.11}$$

Observe that (5.11) always has a solution since $\mu > r$ by assumption. From (5.2) it is also clear that as $\Delta t \to 0$ all the outcomes of the diffusion component will be lower than $\overline{J}$. Therefore, the limiting distribution will include the whole diffusion component and a truncated jump component. The maximum jump outcome in this truncated distribution is obtained from the condition that the distribution is risk neutral, which is expressed in (5.11). As with the upper bound, we can apply the Merton (1976) approach to derive the pde satisfied by the option lower bound, which is given by

$$\left[r - \lambda \mu_J^L \right] S \frac{\partial \underline{C}}{\partial S} - \frac{\partial \underline{C}}{\partial T} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \underline{C}}{\partial S^2} + \lambda E^L [\underline{C}(SJ^L) - \underline{C}(S)] - r\underline{C} = 0 \tag{5.12}$$

with terminal condition $\underline{C}_T = C(S_T, T) = \max\{S_T - K, 0\}$. The solution of (5.12) can be obtained in closed form only when the jump amplitudes are fixed, since even when the jumps are normally distributed, the lower bound jump distribution is truncated.

We present in Table 5.1 and Figure 5.1 estimates of the bounds under a jump-diffusion process for an at-the-money option with K = 100 and maturity T = 0.25 years for varying subdivisions of the time to expiration, and with the following annual parameters: $r = 3\%, \mu = 5\%$ to $9\%, \sigma = 10\%, \lambda = 0.3, \mu_J = -0.05, \sigma_J = 7\%$. The jump-diffusion process was approximated by a 300-time step tree built according to the method introduced by Amin (1993). The bounds were computed by taking the discounted expectation of the payoff under the time-varying risk neutral probabilities of (3.1) applied to subtrees. The risk neutral price is the Merton (1976) price for this process.

Table 5.1

Number of periods	μ=0.09		μ=0.07		μ=0.05		Risk Neutral
	Upper	Lower	Upper	Lower	Upper	Lower	
1	3.5124	1.9731	3.1862	2.0658	2.8763	2.1712	2.3957
2	3.5112	2.0480	3.1855	2.1229	2.8759	2.2084	2.3957
3	3.5107	2.0902	3.1851	2.1562	2.8756	2.2311	2.3957
4	3.5098	2.1182	3.1844	2.1781	2.8753	2.2460	2.3957
5	3.5067	2.1380	3.1824	2.1948	2.8741	2.2574	2.3957
6	3.5022	2.1542	3.1794	2.2061	2.8724	2.2658	2.3957
10	3.4809	2.1932	3.1658	2.2393	2.8646	2.2877	2.3957
12	3.4696	2.2075	3.1583	2.2484	2.8604	2.2951	2.3957
15	3.4523	2.2224	3.1466	2.2613	2.8540	2.3027	2.3957
20	3.4231	2.2401	3.1272	2.2733	2.8431	2.3134	2.3957
25	3.3939	2.2540	3.1077	2.2853	2.8319	2.3184	2.3957
30	3.3645	2.2607	3.0880	2.2932	2.8205	2.3252	2.3957
50	3.2496	2.2890	3.0106	2.3131	2.7757	2.3367	2.3957
60	3.1935	2.2939	2.9727	2.3188	2.7537	2.3431	2.3957
75	3.1127	2.2973	2.9176	2.3219	2.7215	2.3471	2.3957
100	2.9951	2.3180	2.8355	2.3335	2.6726	2.3491	2.3957
150	2.8329	2.3303	2.7186	2.3471	2.6005	2.3635	2.3957
300	2.6299	2.3655	2.5680	2.3722	2.5047	2.3788	2.3957

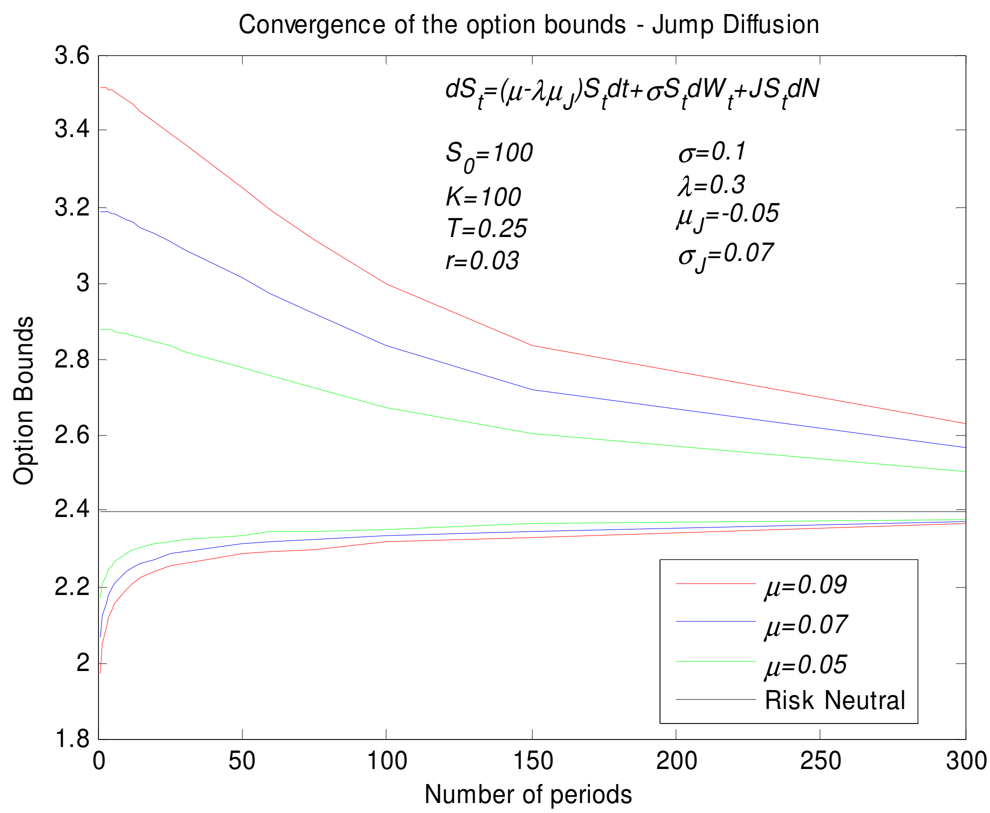

Figure 5.1

The results shown in Table 5.1 show a maximum spread between bounds of about 10.4%[31], a spread that is an increasing function of μ. Note that the range of values of μ implies a risk premium range from 2% to 7%, a range that covers what most people would consider the appropriate value of such a premium; the corresponding width of the bounds ranges from 4.9% to 10.5%. It is important to note that this range of allowable option prices in the stochastic dominance approach is the exact counterpart of the inability of the "traditional" arbitrage-based approaches to produce a single option price for jump diffusion processes, even when the latter have been augmented in this case by general equilibrium considerations. Indeed, the exact option prices under jump diffusion derived in the well-known studies of Bates (1991), Amin and Ng (1993) and Amin (1993) may not depend on μ, but they are all functions of the risk aversion parameter of the Constant Proportional Risk Aversion (CPRA) utility function of consumption used in the derivations; see, for instance, equation (27) of Amin and Ng (1993), or equation (33) of Amin (1993). Further, the assumed monotonicity of the state-contingent discount factors of the stochastic dominance approach in an elementary discrete time period also holds in the combination of jump diffusion asset dynamics and CPRA utility of consumption used in the more traditional approaches. Most important, the range of risk aversion parameter values that have appeared in the literature is extremely wide, ranging from 1 (logarithmic utility) to as high as 30.[32] It can be easily verified that at least for the parameter values given in the above example such a wide range of risk aversion parameter values induces a much larger spread of call option prices than the one arising out of the chosen values of μ in Table 5.1. In other words, the stochastic dominance option bounds are not only a more general approach to option pricing than general equilibrium based on the CPRA form of the utility function, but it can also produce a tighter range of admissible option prices out of the same basic data.

6. Option Bounds in Continuous Time and Incomplete Markets II: Stochastic Volatility Processes

In this section we examine the case when the underlying asset follows a multivariate diffusion, as in the stochastic volatility models of Hull and White (1987), Wiggins (1987), Stein and Stein (1991), Heston (1993), and many others. In these models the asset dynamics are given by equations (4.8)-(4.9), in which the vector x_t of state variables is two-dimensional, consisting of the underlying stock price S_t and of the asset volatility, which is denoted in this section by x_t. Most studies have considered simplified versions of asset dynamics, of the following general form

$$\begin{aligned}&\frac{dS}{S} = \mu(S_t)dt + \sqrt{x}dW_1,\\ &dx = m(x)dt + s(x)dW_2, \quad dW_1 dW_2 = \rho(x)dt\end{aligned} \tag{6.1}$$

[31] The spread is much lower for in-the-money options and reaches about 15% for the out-of-the money options.
[32] See the survey article by Kotcherlakota (1996).

The discrete version of (6.1) can be written as follows

$$
\begin{aligned}
&(S_{t+\Delta t} - S_t)/S_t \equiv z_{t,t+\Delta t} = \mu(S_t)\Delta t + \sqrt{x_t}\varepsilon_{t+\Delta t}\sqrt{\Delta t}, \\
&x_{t+\Delta t} - x_t = m(x_t)\Delta t + s(x_t)\eta_{t+\Delta t}\sqrt{\Delta t}
\end{aligned}
\tag{6.2}
$$

where η_t is an error term of mean 0 and variance 1, and with correlation $\rho(x_t)$ between $\varepsilon_{t+\Delta t}$ and $\eta_{t+\Delta t}$. Since this is a special case of (4.8), Lemma 1 holds and the limiting form of (6.2) is, indeed, (6.1).

The most common form of (6.1) used in option valuation is when the dynamic equation for the volatility is a mean-reverting square root process and the correlation $\rho(x)$ is a constant ρ. A key issue in option pricing under stochastic volatility is the price of the volatility risk that enters into the pde for the option price.[33] In some models such as Hull and White (1987) and Wiggins (1987) the price of the volatility risk has been assumed equal to zero, while in others such as Heston (1993), Bates (1996) and Nandi (1998) the volatility risk has a non-zero price.[34] In all these models the functional form of the volatility risk price has been chosen in order to produce a closed form solution for the option price.

In this section we examine the stochastic dominance bounds arising out of the discretization (6.2) of the stochastic volatility model (6.1) under various assumptions about the market price of volatility risk. We first show that under the assumption of a zero price for the volatility risk both stochastic dominance bounds are available in a recursive closed form and converge to the same value for any type of asset dynamics given by (6.1)-(6.2). Next we relax the volatility risk assumption and examine the stochastic dominance bounds for the special case of CPRA utilities and a restriction on the asset dynamics. We are able to derive recursive expressions for either a lower or an upper bound on option prices, depending on the assumed range of the risk aversion parameter, which is respectively restricted to either the $(0,1]$ or the $[1,\infty)$ subset . We also show that if the risk aversion parameter is not restricted then both bounds assume trivial values even for such a simple case as CPRA utilities.

For the asset dynamics given by (6.1) it can be shown[35] that the option value $C(S_t, x_t)$ is convex in S_t for any x_t. We also know from Merton (1973) that it is also increasing in x_t for any S_t. Under such conditions consider the discretization (6.2) and let $Z(x_t)$ denote the stock returns at $t+\Delta t$, with $C(S_t Z(x_t), x_{t+\Delta t})$ being the corresponding option price, which is convex in $S_t Z$ for any $x_{t+\Delta t}$. Define by $\{Y_{t+\Delta t}\}$ the *conditional* state-contingent discount factors for the returns given the future volatility $x_{t+\Delta t}$, which are monotonically ordered. Let also $\{y_{t+\Delta t}\}$ denote the state-contingent discount factors for

[33] See, for instance, equation (4) in Hull and White (1987) or equations (6) and (7) in Heston (1993).

[34] See the comments by Lamoureux and Lastrapes (1993), p. 9.

[35] See Theorem 2 of Bergman *et al* (1996).

the volatility state variable. We then have

$$C(S_t, x_t) = e^{-r\Delta t} E[y_{t+\Delta t} E[Y_{t+\Delta t} C(S_t Z(x_t), x_{t+\Delta t}) | x_{t+\Delta t}] | x_t], \tag{6.3}$$

where the expectations are under the $P-$distribution. If the volatility risk is not priced then $y_{t+\Delta t} = 1$ and we have the following result, proven in the appendix

Proposition 5: *When the dynamics of the underlying asset follow the stochastic volatility model given by (6.1) and the volatility risk is not priced independently then the option price $C(S_t, x_t)$ corresponding to the discretization (6.2) is at any time $t \in [0, T-1]$ bound by the following convex recursive expressions*

$$\bar{C}(S_{T-\Delta t}, x_{T-\Delta t}) = e^{-r\Delta t} E^{U_{T-\Delta t}}[(SZ(x_{T-\Delta t}) - K)^+], \quad \underline{C}(S_{T-\Delta t}, x_{T-\Delta t}) = e^{-r\Delta t} E^{L_{T-\Delta t}}[(SZ(x_{T-\Delta t}) - K)^+]$$

$$\begin{aligned} &\bar{C}(S_t, x_t) = e^{-r\Delta t} E[E^{U_{t+\Delta t}}[\bar{C}(S_t Z(x_t), x_{t+\Delta t}) | x_{t+\Delta t}] | x_t], \\ &\underline{C}(S_t, x_t) = e^{-r\Delta t} E[E^{L_{t+\Delta t}}[\underline{C}(S_t Z(x_t), x_{t+\Delta t}) | x_{t+\Delta t}] | x_t] \end{aligned} \quad \textit{for } t \le T\text{-}1, \tag{6.4}$$

where the superscripts denote expectations taken with respect to the risk neutral distributions U and L of section 2, which are now conditional on the volatility x_t. Further, Propositions 1 and 2 apply to these bounds as well, which tend to a common value.

Table 6.1 below shows in panel (a) the bounds for various degrees of moneyness *K/S*, evaluated recursively from (6.4) for the following special case of (6.2), the mean-reverting stochastic volatility process, originally presented by Heston (1993), for 20 time periods to option expiration

$$\begin{aligned} &\frac{dS}{S} = \mu dt + \sqrt{x} dW_1, \\ &dx = \kappa(\theta - x)dt + s\sqrt{x} dW_2, \quad dW_1 dW_2 = \rho dt \end{aligned} \tag{6.5}$$

The Heston value in the table denotes the limiting form of the bounds as time becomes continuous, evaluated by the characteristic function approach presented in Heston (1993).[36] We also show in panel (b) and in Figure 1 the value of the bounds for at-the-money options, for T = 0.25 and for various numbers of subdivisions of the time to expiration. We choose again K = 100 and T = 1/12 or 0.25, with an annual mean return μ equal to 0.07 and initial volatility $\sqrt{x_0}$ equal to 0.1. The mean reversion coefficient for the variance equation is $\kappa = 2$, while the long run return variance is $\theta = 0.01$ and the volatility of the variance $s = 0.1$. The riskless rate of interest is 5% and the correlation of the two Brownian motions is $\rho = -0.5$. The numerical method used to compute the

[36] In applying the characteristic function approach we assumed a market price of risk equal to zero.

bounds is described briefly in the appendix.

Table 6.1

(a)

	T = 1/12			T = 0.25		
K/S	Lower bound	Heston value	Upper bound	Lower bound	Heston value	Upper bound
0.95	5.425	5.426	5.426	6.413	6.418	6.424
1	1.366	1.367	1.372	2.667	2.667	2.686
1.05	0.0762	0.076	0.0778	0.690	0.690	0.705

(b)

Number of subdivisions	Lower bound	Heston value	Upper bound
5	2.646	2.667	2.683
10	2.653	2.667	2.679
15	2.663	2.667	2.684
20	2.667	2.667	2.686
25	2.667	2.667	2.684
30	2.667	2.667	2.683

As the table values and Figure 6.1 show, the performance of the bounds is excellent, with a miniscule distance between bounds and with a speedy convergence to the known value.

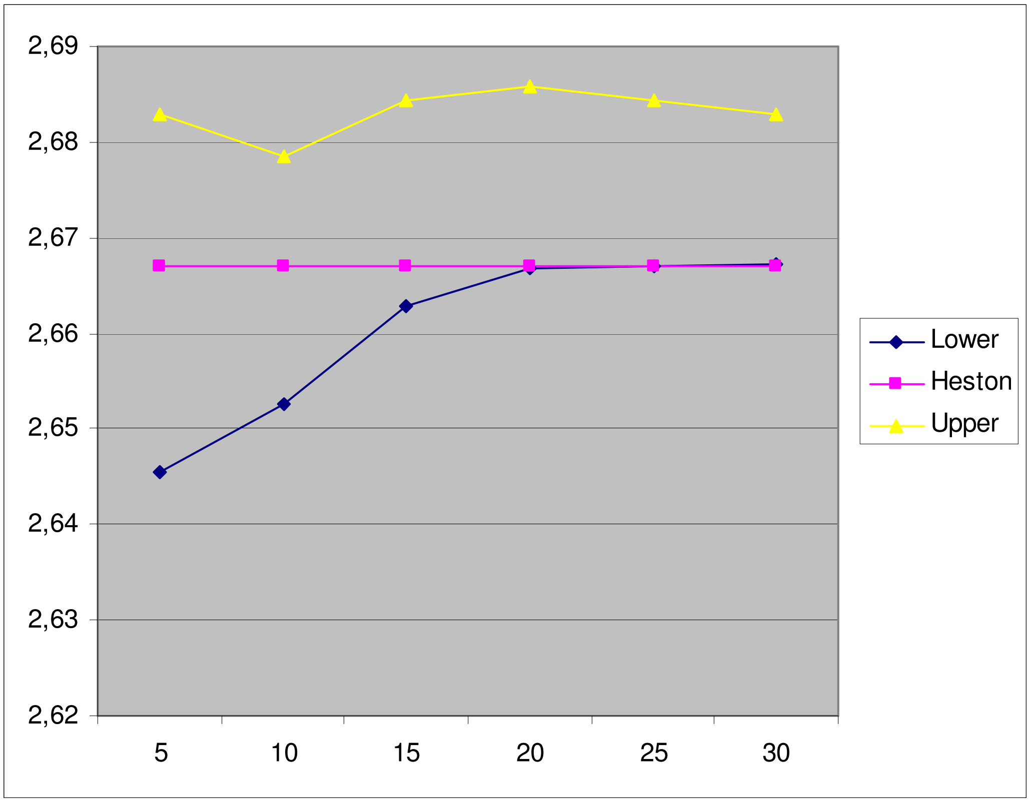

Figure 6.1

Unfortunately it is not possible to generalize the results of Proposition 5 in order to include all possible forms of pricing of the volatility risk. This can be seen very easily from (6.3) and (6.4): a specific price of the volatility risk corresponds to a particular assignment of the state-contingent discount factors $\{y_{t+\Delta t}\}$, which must be non-negative and sum up to 1. If we assume, as is normally the case, that the option value is an increasing function of the volatility x_t then it is clear that without any further assumptions on the state-contingent discount factors $\{y_{t+\Delta t}\}$ the upper and lower bounds on the option prices would correspond respectively to the highest and lowest values of the volatility.[37]

It is possible to apply a variant of the recursive expressions (6.4) in order to find bounds on option values in the presence of non-zero volatility risk price, but only if we restrict the investor preferences and/or the asset dynamics. Suppose, for instance, that the utility

---

[37] Note, however, that the recursive expressions in (6.4) are sufficiently general to accommodate the systematic volatility risk results of Heston (1993) and Amin and Ng (1993, Proposition 2), provided the asset dynamics equations (6.1) are modified to incorporate the parameter(s) representing the price λ of the volatility risk. For instance it suffices to change the drift term in the second equation in (6.5) from $(\kappa\theta - kx)dt$ to $(\kappa\theta - (k + \lambda)x)dt$ and apply Proposition 5 to these new asset dynamics in order to reproduce the Heston (1993) results; see his equation (27). Nonetheless, the market price of risk must still be determined outside the stochastic dominance model and is specific to the assumptions about the form of the investor utility function and the asset dynamics.

of wealth function is of the CPRA type, of the form $u(w)=\frac{w^{1-\alpha}}{1-\alpha}$, $\alpha>0$, where w denotes the investor wealth. Since by assumption the underlying stock is the only risky asset in the investor portfolio, it was shown in Merton (1971) that the indirect utility is a *separable* function of two products, one of which contains the wealth and the other the state variables, which in our case consist of the volatility x_t. Since wealth is also proportional to the stock price S_t, the indirect utility $H(S_t,x_t,t)$ has the following form

$$H(S,x,t)=X(x,t)\frac{S^{1-\alpha}}{1-\alpha}. \tag{6.6}$$

We wish to identify conditions under which the pricing kernel $\frac{H_S}{E(H_S)}$, where $H_S\equiv\frac{\partial H}{\partial S}$, would be monotone in S; note that there are no closed-form expressions for the function $X(x,t)$. At time t the capital market equilibrium in the discretized model yields

$$\frac{E[X(x_{t+\Delta t},t+\Delta t)S_{t+\Delta t}^{\gamma 1-\alpha}|S_t,x_t]}{E[X(x_{t+\Delta t},t+\Delta t)S_{t+\Delta t}^{-\alpha}|S_t,x_t]}=e^{r\Delta t}, \tag{6.7}$$

and the price of the option is given by

$$\begin{aligned}C(S_t,x_t)&=\frac{E[H_S(S_{t+\Delta t},x_{t+\Delta t},t+\Delta t)C(S_{t+\Delta t},x_{t+\Delta t})|S_t,x_t]}{E[H_S(S_{t+\Delta t},x_{t+\Delta t},t+\Delta t)|S_t,x_t]}=\\&=\frac{E[X(x_{t+\Delta t},t+\Delta t)S_{t+\Delta t}^{-\alpha}C(S_{t+\Delta t},x_{t+\Delta t})|S_t,x_t]}{E[X(x_{t+\Delta t},t+\Delta t)S_{t+\Delta t}^{-\alpha}|S_t,x_t]}.\end{aligned} \tag{6.8}$$

A key determinant of the applicability of the stochastic dominance approach to the pricing of volatility risk is the shape of the functions

$$E[X(x,t)S^{-\alpha}|S]\equiv\psi(S.t),\quad E[C(S,x)|S]\equiv\phi(S) \tag{6.9}$$

which depend on the risk aversion parameter α and the dynamics (6.1)-(6.2), especially the sign of the correlation $\rho(x)$. Although there is little that can be said about these functions, we can obtain some results in the special case $\rho(x)=0$ for all x. This is expressed by the following result, proven in the appendix.

Proposition 6: *When the dynamics of the underlying asset follow the stochastic volatility model given by (6.1), the investor utility function is of the CPRA type with risk aversion parameter α, and the correlation coefficient $\rho(x)$ is zero for all x then the option price $C(S_t,x_t)$ corresponding to the discretization (6.2) is at any time $t\in[0,T-1]$ bound by the following convex recursive expressions*

(I) *When* $\alpha \geq 1$ *then* $C(S_t, x_t) \geq \underline{C}(S_t, x_t)$ *, where* $\underline{C}(S_t, x_t)$ *is given by (6.4)*

(II) *When* $\alpha \in (0,1]$ *then* $C(S_t, x_t) \leq \bar{C}(S_t, x_t)$ *, where* $\bar{C}(S_t, x_t)$ *is given by (6.4)*

Unfortunately it is not possible to find an upper (lower) bound for the option price in Case (I) (Case (II)). As it can be easily seen from the proof of Proposition 6, the absence of any restrictions on the shape of the function $X(x,t)$ implies that these bounds don't exist in the absence of additional arbitrary assumptions, even with the simplified asset dynamics assumed here. Nonetheless, Proposition 6 is useful, insofar as it clarifies the results of the well-known stochastic volatility models of Hull and White (1987), Wiggins (1987), and Stein and Stein (1991), which assume the same asset dynamics and zero price for volatility risk. As Wiggins (1987) points out, the only way that one can extract a unique option price for CPRA utility investors is to assume that the representative investor has logarithmic utility, $\alpha = 1$. Most of the empirical evidence in the studies surveyed in Kotcherlakota (1996), however, find that $\alpha > 1$, implying that the stochastic volatility studies that assume zero correlation and zero volatility risk actually estimate lower bounds on the admissible option prices if the volatility risk assumption is relaxed.

It is also not easy to obtain useful results that are not parameter-dependent if we relax the assumption that $\rho(x) = 0$ for all x. To see this consider the case $\alpha \geq 1$, in which case $X(x,t)$ is increasing in x, as shown in Wiggins (1987), noting also that $C(S,x)$ is likewise increasing in x. Then we have, from (6.8) and noting the definitions (6.9)

$$C(S_t, x_t) \geq \frac{E[\psi(S_{t+\Delta t}, t+\Delta t)\phi(S_{t+\Delta t}, t+\Delta t) | S_t, x_t]}{E[\psi(S_{t+\Delta t}, t+\Delta t) | S_t, x_t]}. \tag{6.10}$$

(6.7) now becomes

$$\frac{E[\psi(S_{t+\Delta t}, t+\Delta t) S_{t+\Delta t} | S_t, x_t]}{E[\psi(S_{t+\Delta t}, t+\Delta t) | S_t, x_t]} = e^{r\Delta t}. \tag{6.11}$$

To derive a lower bound from (6.10)-(6.11) we need to assume that $\psi(S.t)$ is monotone decreasing in S, a property that depends on the size of the parameter α and the sign and size of the parameter $\rho(x)$. If such monotonicity is assumed then one can presumably set $\psi(S_{t+\Delta t}, t+\Delta t) \equiv Y_{t+\Delta t}$ and derive recursively a lower bound on $C(S_t, x_t)$ by minimizing the RHS of (6.10) with respect to $Y_{t+\Delta t}$ subject to (6.11).. No closed form solutions would be available, however, since the properties of $\phi(S)$ are similarly parameter-dependent.

7. Option Bounds in Continuous Time and Incomplete Markets III: GARCH models

Last we examine the GARCH model, which has appeared in a number of different formulations in the literature. It is a discrete time model in which the volatility in each successive period depends on the error term of the previous period. Financial trading may or may not take place within each period. If no trading occurs within each period then the market is incomplete and no unique price emerges. One possible approach to “complete” the market is the one originally introduced by Duan (1995), who assumes the existence of a representative investor with a constant proportional risk aversion (CPRA) utility function. This assumption allows the derivation of a local risk neutral valuation operator that transforms the actual return probability distribution into a risk neutral one. We illustrate the derivation of the stochastic dominance bounds for the NGARCH specification used by Duan (1995). In that model the asset dynamics are given by

$$\begin{aligned} \ln\frac{S_{t+1}}{S_t} &= r+\lambda\sqrt{h_{t+1}}-\frac{1}{2}h_{t+1}+\sqrt{h_{t+1}}\varepsilon_{t+1} \\ h_{t+1} &= \beta_0+\beta_1 h_t+\beta_2\varepsilon_t^2 \qquad , \\ \varepsilon_{t+1} &\mid \mathcal{F}_t \sim N(0,1) \end{aligned} \tag{7.1}$$

in which h_{t+1}, the volatility of the innovations ε_{t+1}, is a function of past volatilities and innovations. Note that according to the standard notation used in the GARCH literature h_{t+1} is known at time t. The parameter λ is proportional to the risk premium of the stock.

The value $C(S_t,h_{t+1})$ of an option when the underlying asset follows the dynamics given by (7.1) is given by a discounted recursive expectation with a pricing kernel $\{Y_{t+1}\}$

$$\begin{aligned} C(S_{T-1},h_t) &= R^{-1}E[Y_{T-1}(S_T-K)^+] \\ C(S_t,h_{t+1}) &= R^{-1}E[Y_{t+1}C(S_{t+1},h_{t+2})\big|S_t,h_{t+1}] \end{aligned} \tag{7.2}$$

While the convexity of $C(S_t,h_{t+1})$ as a function of S_t is a consequence of Merton’s (1973) Theorem 10, the function $C(S_{t+1},h_{t+2}\mid S_t,h_{t+1})$ is not necessarily convex in S_{t+1}, since the error term ε_{t+1} enters independently into both the price S_{t+1} and the volatility h_{t+2}. Although we cannot find bounds on *all* option prices corresponding to the asset dynamics given by (7.1), we can prove the following result, proven in the appendix.

Proposition 7: *When the dynamics of the underlying asset follow the GARCH model given by (7.1) then there exist option prices* $C(S_t,\varepsilon_t)$ *corresponding to the valuation model (7.2) that are at any time* $t\in[0,T-1]$ *bound by the following convex recursive expressions*

$$\overline{C}(S_{T-1},h_T)=R^{-1}E^{U_{T-1}}[(S_T-K)^+],\quad \underline{C}(S_{T-1},h_T)=R^{-1}E^{L_{T-1}}[(S_T-K)^+],$$

$$\left.\begin{aligned}\overline{C}(S_t,h_{t+1})&=R^{-1}E^{U_t}[\overline{C}(S_{t+1},h_{t+2}\mid S_t,h_{t+1})],\\ \underline{C}(S_t,h_{t+1})&=R^{-1}E^{L_t}[\underline{C}(S_{t+1},h_{t+2}\mid S_t,h_{t+1})]\end{aligned}\right.\quad \textit{for } t\le T\textit{-1}. \tag{7.3}$$

where the superscripts denote expectations taken with respect to the risk neutral distributions U and L of section 2.

In the numerical results presented below the error term ε_t is approximated by a multinomial variable of mean 0 and variance 1, as in the Markov chain approach of Duan and Simonato (2001). The superscript RN denotes the risk neutral Duan (1995) and Duan and Simonato (2001) estimates of the option value. We expect the following relationship between the stochastic dominance bounds and the risk neutral option price:

$$\underline{C}_t \le C_t^{RN} \le \overline{C}_t.$$

Table 7.1 below shows European call and put option bounds, as well as the option price corresponding to the Duan (1995) model, for various values of the time to expiration and the moneyness of the option. The partition of the time interval to option expiration is equal to one day for all times to expiration. The parameters are $S=50, r=0.05$, and for $T=3$ months (90 partitions corresponding to one day each) the parameters of the NGARCH process are $\beta_0=0.00001, \beta_1=0.8, \beta_2=0.1, \theta=0.3$ and $\lambda=0.01$.

Table 7.1

GARCH Option Prices

K/S	*Time (months)*	*Call Price* $\underline{C}$	C^{RN}	$\overline{C}$	*Put Price* $\underline{P}$	P^{RN}	$\overline{P}$
0.9	1	5.2605	5.2616	5.2791	0.0715	0.0765	0.0878
	3	5.9497	5.9726	6.0162	0.3916	0.4122	0.4552
	6	6.9146	6.9572	7.0288	0.8046	0.8421	0.9161
1	1	1.287	1.308	1.339	1.0773	1.102	1.1269
	3	2.4116	2.4563	2.5206	1.7914	1.8337	1.8976
	6	3.6104	3.6747	3.7709	2.3769	2.4362	2.5347
1.1	1	0.0595	0.0691	0.0744	4.829	4.8424	4.8416
	3	0.6081	0.6468	0.6874	4.9257	4.9621	5.0022
	6	1.5425	1.6072	1.6909	5.1855	5.2452	5.3312

The table shows that the bounds perform very well. In all but one case the risk neutral price lies within the bounds and the only exception, the one-month in-the-money put, is equal to the upper bound within the limits of the accuracy of our computations.

An alternative assumption that may be used in deriving risk neutral option prices under a GARCH returns structure is to assume that there is continuous trading within each successive GARCH period. This is the approach taken by Kallsen and Taqqu (1998). Each GARCH subperiod is partitioned in intervals of length Δt, and the returns for each such interval are then given by (4.8) with the corresponding constant volatility, and the corresponding stochastic dominance bounds are given by (4.10); the volatility varies as we shift from one GARCH period to the subsequent one. As the partition Δt within each GARCH period becomes progressively finer, with the number of GARCH periods remaining the same, the two bounds also tend to the same value.[38] To see this we note that the GARCH model of (7.1) becomes now

$$\frac{dS_t}{S_t} = (r + \lambda\sqrt{h_t})dt + \sqrt{h_t}\,dW, \tag{7.4}$$

where

$$h_t = \begin{cases} h_0 & \text{if } 0 \le t < 1 \\ \beta_0 + \beta_1 h_{[t]-1} + \beta_2 (\ln \dfrac{S_{[t]}}{S_{[t]-1}} - r + \dfrac{1}{2} h_{[t]} - \lambda\sqrt{h_{[t]-1}} - c\sqrt{h_{[t]-1}})^2 & \text{if } t \ge 1 \end{cases} \tag{7.5}$$

In the above formula, $[t]$ is the largest integer number that is less than t. Suppose there are n GARCH periods till expiration of the option. Then we can use induction to prove that the two stochastic dominance bounds coincide. At any time t = n-1 the two option bounds obviously coincide, since we have a simple diffusion. Now we can apply induction, by assuming that the bounds coincide for any time $t = \tau$, where $\tau < t-1$, and then use Lemma 1 and Propositions 1 and 2 to demonstrate that the bounds would also coincide for any time $t \in [\tau - 1, \tau)$. The common limit of the two bounds is the local risk neutral price of Duan (1995).

8. Conclusions

In this paper we have presented a new approach to option pricing, which we have termed the stochastic dominance approach. This approach derives two bounds on allowable option prices dependent on the entire distribution of underlying asset returns. The distribution can be of any type, but the contingent claims are restricted to options with convex payoffs. We show that the two bounds are discounted payoff expectations under two risk neutral transformations of the original asset dynamics which remain *the same* under all types of asset dynamics.

We then examined the convergence of the discrete time option bounds derived by stochastic dominance methods in a multiperiod context as trading becomes progressively

[38] Alternatively, the length of the GARCH period itself could tend to zero at the limit. In such a case, the NGARCH process tends to a stochastic volatility model, for which the two bounds coincide. See Duan (1997).

more dense, under a variety of assumptions about the limiting distribution of the underlying asset returns. We found that this stochastic dominance approach nests virtually the entire set of option prices available in the literature under a variety of alternative methods, including arbitrage and general equilibrium. Specifically, they nest all the models where the distribution of underlying asset depends on a single random factor, as well as the models in which this same distribution depends on multiple factors, provided the pricing operator depends only on a single factor. They fail only whenever these multiple factors affect also the pricing operator over and above the effect that they may have on the price of the underlying asset, although the stochastic dominance approach still provides valuable insights in such cases as well.

The stochastic dominance approach depends crucially on the convexity of the option payoffs, which makes it suitable for "plain vanilla" call and put options, but its extension to contingent claims with more complex payoff patterns is uncertain. It is, in principle, possible to apply the single period linear programming approach of section 2 to non-convex payoffs, but the resulting contingent claim bounds would no longer have a closed form solution under general conditions. They would, therefore, not be easily amenable either to multiperiod formulations or to the limiting techniques applied in this paper. Similar difficulties are expected to arise if the pricing operator contains random factors that cannot be represented by the price of the underlying asset. Extensions of the stochastic dominance approach to incorporate these additional complications should be a major objective of future research.

On the other hand, there are two major advantages of the stochastic dominance approach over alternative derivatives pricing methods. The first one is that it does produce useful results in the presence of market frictions such as transaction costs, in sharp contrast to the arbitrage approach.[39] The second one is that it is not necessary to know the stochastic process governing the evolution of the price of the underlying asset in order to price the derivative, as long as an empirical distribution represented by a histogram of possible future values (or returns) of the asset is available. Such an empirical distribution is sufficient to derive the risk neutral *U*- and *L*-distributions that define the option bounds. The common derivation of these pricing distributions would presumably minimize any possible errors in option price stemming from the choice of the wrong model. The empirical implications of this second advantage should form the object of future research.

[39] See Constantinides and Perrakis (2002, 2007).

References

Amin, Kaushik I., 1993, “Jump Diffusion Option Valuation in Discrete Time”, *Journal of Finance*, 48, 1833-1863.

Amin, Kaushik I., and Victor K. Ng, 1993, “Option Valuation with Systematic Stochastic Volatility”, *The Journal of Finance* 48, 881-910.

Bakshi, Gurdip S., Charles Cao, and Zhiwu Chen, 1997, “Empirical Performance of Alternative Option Pricing Models”, *The Journal of Finance* 52, 2003-2049.

Bates, D. S. (1991), “The Crash of ’87: Was it Expected? The Evidence from Option Markets”, *Journal of Finance*, 46, 1009-1044.

Bates, David, 1996, “Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options”, *Review of Financial Studies* 9, 69-107.

Bergman, Yaacov Z., Bruce D. Grundy, and Zvi Wiener, 1996, “General Properties of Option Prices”, *The Journal of Finance* 51, 1573-1610.

Black, Fischer, and Myron Scholes, 1973, “The Pricing of Options and Corporate Liabilities”, *Journal of Political Economy* 81, 637-659.

Bogle, J. C., 2005, "The Mutual Industry 60 Years Later: For Better or Worse?" *Financial Analyst Journal* 61, 15-24.

Brennan, Michael, 1979, “The Pricing of Contingent Claims in Discrete-Time Models”, *The Journal of Finance* 34, 53-68.

Constantinides, G. M., J. Jackwerth and S. Perrakis, 2007, “Mispricing of S&P 500 Index Options”, forthcoming in the *Review of Financial Studies*.

Constantinides, G. M., and S. Perrakis, 2002, "Stochastic Dominance Bounds on Derivative prices in a Multiperiod Economy with Proportional Transaction Costs", *Journal of Economic Dynamics and Control* 26, 1323-1352**.**

Constantinides, G. M., and S. Perrakis, 2007, "Stochastic Dominance Bounds on American Option Prices in Markets with Frictions", *Review of Finance* 11, 71-115.

Cox, John, and Stephen A. Ross, 1976, “The Valuation of Options for Alternative Stochastic Processes”, *Journal of Financial Economics* 3, 145-166.

Cox, John C., and M. Rubinstein, 1985, *Options Markets*, Prentice Hall, Englewood Cliffs, N. J.

Duan, Jin-Chuan, 1995, “The GARCH Option Pricing Model, *Mathematical Finance* 5, 13-32.

Duan, Jin-Chuan, 1997, “Augmented GARCH(p,q) process and its diffusion limit”, *Journal of Econometrics* 79, 97-127.

Duan, Jin-Chuan, and Jean-Guy Simonato, 2001, “American option pricing under GARCH using a Markov chain approximation”, *Journal of Economic Dynamics and Control* 25, 1689-1718.

Duffie, Darrell, Jun Pan and Kenneth Singleton, 2000, “Transform Analysis and Asset Pricing for Affine Jump-Diffusions'”, *Econometrica,* 68, 1343-1376.

Ethier, Stewart N., and Thomas G. Kurtz, *Markov Processes*, Wiley, New York, 1986.
Grundy, Bruce D., 1991, “Option Prices and the Underlying Asset’s Return Distribution”, *The Journal of Finance* 46, 1045-1069.

Grundy, B., 1991, “Option Prices and the Underlying Asset’s Return Distributions”, *Journal of Finance* 46, 1045-1069

Harrison, John M., and David M. Kreps, 1979, “Martingales and Arbitrage in Multiperiod Securities Markets”, *Journal of Economic Theory* 20, 381-408.

Harrison, John M., and Stanley Pliska, 1981, “Martingales and Stochastic Integrals in the Theory of Continuous Trading”, *Stochastic Processes and their Applications* 11, 215-260.

Heston, Steven, 1993, “A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options”, *Review of Financial Studies* 6, 327-343.

Hull, John, and Alan White, 1987, “The Pricing of Options on Assets with Stochastic Volatilities”, *Journal of Finance* 42, 281-300.

Jackwerth, J. C., 2004, *Option-Implied Risk-Neutral Distributions and Risk Aversion,* ISBN 0-943205-66-2, Research Foundation of AIMR, Charlotteville, USA.

Johnson, H., and D. Shanno, 1987, “Option Pricing when the Variance is Changing”, *Journal of Financial and Quantitative Analysis*, 22, 143-151.

Kallsen, J., and M. Taquu, 1998, “Option Pricing in ARCH-Type Models”, *Mathematical Finance* 8, 13-26.

Kocherlakota, N. R., 1996, "The Equity Premium: It’s Still a Puzzle." *Journal of Economic Literature* 34, 42-71.

Lamoureux, Christopher G., and William D. Lastrapes, 1993, “Forecasting Stock-Return

Variance: Toward an Understanding of Stochastic Implied Volatilities", *Review of Financial Studies* 6, 293-326.

Levy, Haim, 1985, "Upper and Lower Bounds of Put and Call Option Value: Stochastic Dominance Approach", *The Journal of Finance* 40, 1197-1217.

Masson, J. and S. Perrakis, 2000, "Option Pricing Bounds and the Pricing of the Volatility Smile", *Review of Derivatives Research* 4, 1 (2000), 29-53.

Merton, Robert C., 1971, "Optimum Concumption and Portfolio Rules in a Continuous Time Model", *Journal of Economic Theory* 3, 373-413.

Merton, Robert C., 1973, "Theory of Rational Option Pricing", *Bell Journal of Economics and Management Science* 4, 141-183.

Merton, Robert C., 1976, "Option Pricing when the Underlying Stock Returns are Discontinuous", *Journal of Financial Economics* 3, 125-144.

Merton, Robert C., 1982, "On the Mathematics and Economics Assumptions of Continuous-Time Models", in *Essays in Honor of Paul Cootner*, Prentice Hall, Englewood Cliffs, NJ, 1982.

Merton, R. C., 1989, "On the Application of the Continuous-Time Theory of Finance to Financial Intermediation and Insurance." *The Geneva Papers on Risk and Insurance* 14, 225-261.

Merton, Robert C., *Continuous Time Finance*, Basil Blackwell, Cambridge, MA, 1992.

Nandi, S., 1998, "How Important is the Correlation Between Returns and Volatility? Empirical Evidence from Pricing and Hedging in the S&P 500 Index Options Market", *Journal of Banking and Finance*, 22, 589-610.

Perrakis, Stylianos, 1986, "Option Bounds in Discrete Time: Extensions and the Pricing of the American Put", *Journal of Business* 59, 119-141.

Perrakis, Stylianos, 1988, "Preference-free Option Prices when the Stock Returns Can Go Up, Go Down or Stay the Same", in Frank J. Fabozzi, ed., *Advances in Futures and Options Research*, JAI Press, Greenwich, Conn.

Perrakis, Stylianos, and Peter J. Ryan, 1984, "Option Bounds in Discrete Time", *The Journal of Finance 39*, 519-525.

Ritchken, Peter H., 1985, "On Option Pricing Bounds", *The Journal of Finance* 40, 1219-1233.

Ritchken, Peter H., and Shyanjaw Kuo, 1988, "Option Bounds with Finite Revision Opportunities", *The Journal of Finance* 43, 301-308.

Soner, H. M., S. E. Shreve and J. Cvitanic, 1995, "There is No Nontrivial Hedging Portfolio for Option Pricing with Transaction Costs." *The Annals of Applied Probability* 5, 327-355.

Stein, E., and J. C. Stein, 1991, "Stock Price Distributions with Stochastic Volatility: an Analytic Approach", *Review of Financial Studies* 4, 727-752.

Strook, Daniel W., and S.R. Srinivasa Varadhan, *Multidimensional Diffusion Processes*, Springer-Verlag, Berlin, 1979.

Appendix

A. Proof of Lemma 1

The proof is similar to the one used by Merton (1982), the only difference being that $\varepsilon_{t+\Delta t}$ is now a bounded continuous random variable rather than a multinomial discrete one. Denote $Q_t(\delta)$ the conditional probability that $|x_{t+\Delta t} - x_t| > \delta$, given the information available at time t. Since $\varepsilon_{t+\Delta t}$ is bounded, define $\bar{\varepsilon} = \max|\varepsilon_{t+\Delta t}| = \max(|\varepsilon_{\min}|, |\varepsilon_{\max}|)$. For any $\delta > 0$, define $h(\delta)$ as the solution of the equation

$$\delta = \mu h + \sigma\bar{\varepsilon}\sqrt{h}.$$

This equation admits a positive solution

$$\sqrt{h} = \frac{-\sigma\bar{\varepsilon} + \sqrt{\sigma^2\bar{\varepsilon}^2 + 4\mu\delta}}{\mu}.$$

For any $\Delta t < h(\delta)$ and for any possible $x_{t+\Delta t}$,

$$|x_{t+\Delta t} - x_t| = |\mu\,\Delta t + \sigma\varepsilon_{t+\Delta t}\sqrt{\Delta t}| < \mu h + \sigma\bar{\varepsilon}\sqrt{h} = \delta$$

so $Q_t(\delta) = \Pr(|x_{t+\Delta t} - x_t| > \delta \equiv 0$ whenever $\Delta t < h$ and hence

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} Q_t(\delta) = 0$$

The Lindeberg condition is thus satisfied. Equations (4.3) and (4.4) are satisfied by the construction of this discrete process, so the diffusion limit of (4.7) is (4.1), QED.

B. Proof of Proposition 1

We shall consider only the case $\mu > r$; the proof for the case $\mu \leq r$ is similar and is omitted. Under the upper bound probability given by (2.7), the returns process becomes

$$z_{t,t+\Delta t} = \begin{cases} z_{t,t+\Delta t} & \text{with probability} \quad 1-Q \\ \min z_{t,t+\Delta t} & \text{with probability} \quad Q \end{cases},$$

where Q is the following probability

$$Q = \frac{E(z) - r\,\Delta t}{E(z) - \min(z_{t,t+\Delta t})}$$

$$= \frac{\mu\,\Delta t - r\,\Delta t}{\mu\,\Delta t - (\mu\,\Delta t + \sigma\varepsilon_{\min}\sqrt{\Delta t})} = -\frac{\mu - r}{\sigma\varepsilon_{\min}}\sqrt{\Delta t}$$

From the definition of $z_{t,t+\Delta t}$ given in (4.8) we get

$$z_{t,t+\Delta t} = \mu(x_t)\Delta t + \sigma(x_t)\sqrt{\Delta t}\begin{cases}\varepsilon_{t,t+\Delta t} & \text{with probability} \quad 1-Q \\ \varepsilon_{\min} & \text{with probability} \quad Q\end{cases} \tag{G.1}$$

The random component of the returns in (A1) has a bounded continuous distribution, so the upper bound process satisfies the Lindeberg condition. The upper bound distribution (A1) has the mean

$$E_t^U[z_{t,t+\Delta t}] = \mu\Delta t + (1 + \frac{\mu - r}{\sigma\varepsilon_{\min}}\sqrt{\Delta t})(\sigma\sqrt{\Delta t})E_t[\varepsilon_{t,t+\Delta t}]$$

$$-\frac{\mu - r}{\sigma\varepsilon_{\min}}\sqrt{\Delta t}(\sigma\sqrt{\Delta t})\varepsilon_{\min}$$

$$= \mu\Delta t - \frac{\mu - r}{\sigma\varepsilon_{\min}}\sqrt{\Delta t}(\sigma\sqrt{\Delta t})\varepsilon_{\min} = r\Delta t$$

Its variance is

$$\mathrm{Var}_t^U[z_{t,t+\Delta t}] = \sigma_t^2(x_t)\Delta t\left[\left(1 + \frac{\mu_t - r}{\sigma_t\varepsilon_{\min}}\sqrt{\Delta t}\right)\mathrm{Var}_t[\varepsilon_{t,t+\Delta t}]\right.$$

$$\left. - \frac{\mu_t - r}{\sigma_t\varepsilon_{\min}}\sqrt{\Delta t}\varepsilon_{\min}^2\right]$$

$$= \sigma_t(x_t)\Delta t\left[1 + \frac{\mu_t - r}{\sigma_t\varepsilon_{\min}}\sqrt{\Delta t} - \frac{\mu_t - r}{\sigma_t\varepsilon_{\min}}\sqrt{\Delta t}\varepsilon_{\min}^2\right]$$

$$= \sigma_t(x_t)\Delta t + o(\Delta t)$$

Consequently, the upper bound process converges weakly to the diffusion (4.11).

C. Proof of Proposition 2

As with the proof of Proposition 1, we shall consider only the case $\mu > r$. Under the probability distribution given by (2.10) for the lower bound the transformed returns process becomes

$$z_{t,t+\Delta t} = \mu(x_t)\Delta t + \sigma(x_t)\hat{\varepsilon}_t\sqrt{\Delta t},$$

where $\hat{\varepsilon}_t$ is a truncated random variable $\{\hat{\varepsilon}_t \mid \varepsilon_t < \overline{\varepsilon}\}$, with $\overline{\varepsilon}$ found from the condition $E^L[z_{t,t+\Delta t}] = r\Delta t$. Since $\hat{\varepsilon}_t$ is truncated from a bounded continuous distribution the Lindeberg condition is satisfied. The risk neutrality of the lower bound distribution implies that

$$\mu\Delta t + \sigma\sqrt{\Delta t}E[\hat{\varepsilon}_t] = r\Delta t,$$

and the mean of $\hat{\varepsilon}_t$ is

$$E[\hat{\varepsilon}_t] = -\frac{\mu - r}{\sigma}\sqrt{\Delta t} \tag{G.2}$$

Since this random variable is drawn from a distribution that is truncated from the distribution D of ε_t we get

$$E[\hat{\varepsilon}_t] = \frac{1}{\Pr(\varepsilon_t < \overline{\varepsilon}_t)}\int_{\varepsilon_{\min}}^{\overline{\varepsilon}_t} \varepsilon_t dD(\varepsilon_t). \tag{G.3}$$

We picked ε_t such that $E_t[\varepsilon_t] = 0$ and we have

$$\int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \varepsilon_t dD(\varepsilon_t) = \int_{\varepsilon_{\min}}^{\overline{\varepsilon}_t} \varepsilon_t dD(\varepsilon_t) + \int_{\overline{\varepsilon}_t}^{\varepsilon_{\max}} \varepsilon_t dD(\varepsilon_t) = 0. \tag{G.4}$$

Then, from (A2)-(A4) we get

$$\begin{aligned}\frac{\mu - r}{\sigma}\sqrt{\Delta t} &= \frac{1}{\Pr(\varepsilon_t < \overline{\varepsilon}_t)}\int_{\overline{\varepsilon}_t}^{\varepsilon_{\max}} \varepsilon_t dD(\varepsilon_t) \\ &\geq \frac{1}{1 - \Pr(\varepsilon_t > \overline{\varepsilon}_t)}\int_{\overline{\varepsilon}_t}^{\varepsilon_{\max}} \overline{\varepsilon}_t dD(\varepsilon_t) \\ &= \frac{\overline{\varepsilon}_t \Pr(\varepsilon_t > \overline{\varepsilon}_t)}{1 - \Pr(\varepsilon_t > \overline{\varepsilon}_t)}.\end{aligned} \tag{G.5}$$

From the last inequality of (A5) we get

$$\Pr(\varepsilon_t > \overline{\varepsilon}_t) \leq \frac{\frac{\mu - R}{\sigma}\sqrt{\Delta t}}{\overline{\varepsilon}_t + \frac{\mu - R}{\sigma}\sqrt{\Delta t}} = O(\sqrt{\Delta t}). \tag{G.6}$$

(A6) implies that as $\Delta t \to 0$ the probability for all $\varepsilon_t > \overline{\varepsilon}$ tends to zero. Therefore, the limit lower bound distribution contains all the possible outcomes of ε_t . This result is used to compute the variance of $\hat{\varepsilon}_t$

$$
\begin{aligned}
Var(\hat{\varepsilon}_t) &= E[\hat{\varepsilon}_t^2] - (E[\hat{\varepsilon}_t])^2 \\
&= \frac{1}{\Pr(\varepsilon_t < \bar{\varepsilon}_t)} \int_{\varepsilon_{\min}}^{\bar{\varepsilon}_t} \varepsilon_t^2 dD(\varepsilon_t) - \left(\frac{\mu - r}{\sigma}\right)^2 \Delta t \\
&= \int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \varepsilon_t^2 dD(\varepsilon_t) - \left(\frac{\mu - r}{\sigma}\right)^2 \Delta t \\
&= 1 - \left(\frac{\mu - r}{\sigma}\right)^2 \Delta t \sim 1,
\end{aligned}
$$

where the third equality applies the conclusion derived from (A6) and the last equality uses the fact that

$$
Var(\varepsilon_t) = \int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \varepsilon_t^2 dD(\varepsilon_t) = 1
$$

It follows that

$$
Var_t^L[z_{t,t+\Delta t}] = \sigma^2 \Delta t + O(\Delta t)^2
$$

The diffusion limit is, therefore, the process described by equation (4.11), QED.

D. Proof of Lemma 2

As shown in the proof of Lemma 1, the first two terms of (5.2) converge to a diffusion. The generator of this diffusion is

$$
\begin{aligned}
\mathcal{A}f &= \lim_{\Delta t \to 0} \frac{E[f(z_{t+\Delta t}, t + \Delta t)] - f(z_t, t)}{\Delta t} \\
&= (\mu_t - \lambda \mu_J) S \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2} \sigma_t^2 S^2 \frac{\partial^2 f}{\partial S^2}.
\end{aligned}
$$

Denote $\mathcal{A}_J^{\Delta t}$ the generator of the discrete process described by (5.2). This generator converges to

$$
\begin{aligned}
\lim_{\Delta t \to 0} \mathcal{A}_J^{\Delta t} f &= \lim_{\Delta t \to 0} \frac{E[f(z_{t+\Delta t}, t + \Delta t)] - f(z_t, t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} (1 - \lambda \Delta t) \frac{E[f(z_{D,t+\Delta t}, t + \Delta t)] - f(z_{D,t}, t)}{\Delta t} \\
&+ \lambda \Delta t \frac{E[f(z_{J,t+\Delta t}, t + \Delta t)] - f(z_{J,t}, t)}{\Delta t} \\
&= (\mu_t - \lambda \mu_J) S \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} + \frac{1}{2} \sigma_t^2 S^2 \frac{\partial^2 f}{\partial S^2} + \lambda E[f(SJ) - f(S)],
\end{aligned} \tag{G.7}
$$

which is the generator of (5.1), QED.[40]

E. Proof of Proposition 3

As with Propositions 1 and 2, we consider the multiperiod discrete time bounds of section 2, obtained by successive expectations under the risk-neutral upper bound distribution. We then seek the limit of this distribution as $\Delta t \to 0$. The probability Q used in equation (2.7) is given by

$$Q = \frac{E(z) - r\Delta t}{E(z) - J_{\min}} = \frac{(\mu - r)\Delta t}{\mu\Delta t - J_{\min}} = \lambda_U \Delta t,$$

where

$$\lambda_U = -\frac{\mu_t - r}{J_{\min}},$$

since the expected return under the subjective probability distribution is

$$E(z_{t+\Delta t}) = (1 - \lambda\Delta t)(\mu - \lambda\mu_J)\Delta t + \lambda\mu_J\Delta t = \mu\Delta t + o(\Delta t).$$

Observe that λ_U is always positive since $J_{\min} < 0$ and $E(z) > r\Delta t$. Hence, the discrete time upper bound process is

$$z_{t,t+\Delta t} = \begin{cases} z_D & \text{with probability} \quad (1 - \lambda\Delta t)(1 - \lambda_U \Delta t), \\ J & \text{with probability} \quad \lambda\Delta t(1 - \lambda_U \Delta t), \\ J_{\min} & \text{with probability} \quad \lambda_U \Delta t. \end{cases}$$

By removing the terms in $o(\Delta t)$, the upper bound process becomes

$$z_{t,t+\Delta t} = \begin{cases} z_D & \text{with probability} \quad 1 - (\lambda + \lambda_U)\Delta t \\ J^U & \text{with probability} \quad (\lambda + \lambda_U)\Delta t \end{cases}, \tag{G.8}$$

where J^U is given by (5.5). This is a mixture of the diffusion component and a jump with intensity $\lambda + \lambda_U$. It can be readily verified that the upper bound process is risk neutral by construction. By Lemma 2, therefore, it converges weakly to a jump-diffusion process with the generator

[40]See for instance Merton (1992) for a discussion on the generators of diffusions and jump processes.

$$\mathcal{A}^U f = \left[r - (\lambda + \lambda_U)\mu_J^U \right] S \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} \\ + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \lambda_U E^U [f(SJ^U) - f(S)]. \tag{G.9}$$

This process, however, corresponds to (5.3), QED.

F. Proof of Proposition 4

The proof is very similar to those of Lemma 2 and Proposition 3. We apply equation (2.10) and observe that, as with the upper bound, the lower bound distribution over $(t, t+\Delta t)$ is a mixture of the diffusion component and a jump of intensity λ and log-amplitude distribution J^L, the truncated distribution $\{J \mid J \le \bar{J}\}$.

$$z_{t,t+\Delta t} = \begin{cases} z_D & \text{with probability} \quad 1 - \lambda\Delta t \\ J^L & \text{with probability} \quad \lambda\Delta t \end{cases}. \tag{G.10}$$

By Lemma 2 this process converges weakly for $\Delta t \to 0$ to a jump-diffusion process with generator

$$\mathcal{A}^L f = \left[r - \mu_J^L \right] S \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} \\ + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \lambda E^L [f(S + J^L) - f(S)]. \tag{G.11}$$

(A11), however, corresponds to (5.10), QED.

G. Proof of Proposition 5

Let $\Delta t = \dfrac{T}{n}$ denote the length of the discrete time period. At $T - \Delta t$ the option price $C(S_{T-\Delta t}, x_{T-\Delta t})$ is clearly bound by the expectations E^U and E^L of the payoff, which are convex functions of $S_{T-\Delta t}$ for any $x_{T-\Delta t}$, as well as functions of $x_{T-\Delta t}$. Assume now that at $t + \Delta t$ the price $C(S_{t+\Delta t}, x_{t+\Delta t})$ is similarly bound by the expressions $\bar{C}(S_{t+\Delta t}, x_{t+\Delta t})$ and $\underline{C}(S_{t+\Delta t}, x_{t+\Delta t})$ given by (6.4), which are convex in $S_t Z(x_t)$ for any $x_{t+\Delta t}$. By the induction hypothesis we have, from (6.3), setting $y_{t+\Delta t} = 1$

$$e^{-r\Delta t} E[E[Y_{t+\Delta t}\, \underline{C}(S_t Z(x_t), x_{t+\Delta t}) \big| x_{t+\Delta t}] x_t] \le C(S_t, x_t) \\ \le e^{-r\Delta t} E[E[Y_{t+\Delta t}\, \bar{C}(S_t Z(x_t), x_{t+\Delta t}) \big| x_{t+\Delta t}] x_t] \tag{A.12}$$

From (A.12) we get, by the assumed convexity of $\bar{C}(S_{t+\Delta t}, x_{t+\Delta t})$ and $\underline{C}(S_{t+\Delta t}, x_{t+\Delta t})$

$$e^{-r\Delta t} E[E^{L_{t+\Delta t}}[\underline{C}(S_t Z(x_t), x_{t+\Delta t}) | x_{t+\Delta t}] x_t] \leq C(S_t, x_t) \leq e^{-r\Delta t} E[E^{U_{t+\Delta t}}[\bar{C}(S_t Z(x_t), x_{t+\Delta t}) | x_{t+\Delta t}] x_t]$$

Observe that the expressions within brackets in both sides of this last expression are convex in S_t for any given $x_{t+\Delta t}$ since they are expectations of functions of $S_t Z(x_t)$ that are convex by the induction hypothesis. Hence, their expectation with respect to $x_{t+\Delta t}$ is also convex. Hence, $C(S_t, x_t)$ is bound by the convex expressions in (6.5), QED.

H. Monte Carlo Estimation of the Proposition 5 Bounds

The bounds in equations (6.4) can be further written as expectations of the final payoff, conditional on the current asset price and the path of the volatility $x_{t+\Delta t}, \ldots x_{T-\Delta T}$ until the expiration of the option.

$$\bar{C}(S_t, x_t) = e^{-r\Delta t} E[E^{U(x_t, x_{t+\Delta t}, \ldots x_{T-\Delta T})}[(S_T - K)^+ | x_{t+\Delta t}, \ldots x_{T-\Delta T}] \mid S_T, x_t],$$
$$\underline{C}(S_t, x_t) = e^{-r\Delta t} E[E^{L(x_t, x_{t+\Delta t}, \ldots x_{T-\Delta T})}[(S_T - K)^+ | x_{t+\Delta t}, \ldots x_{T-\Delta T}] \mid S_T, x_t]$$

The probability measures $U(x_t, x_{t+\Delta t}, \ldots x_{T-\Delta T})$ and $L(x_t, x_{t+\Delta t}, \ldots x_{T-\Delta T})$ are convolutions of the single-period conditional bound probabilities over each path of the volatility. For a given discretization of the underlying process, the single-period probabilities are given by the expressions (2.4) and (2.5). The availability of the conditional bound probabilities in closed form enables the computation of the bounds by Monte Carlo simulation, using modified forms of the expressions given in Hull and White (1987, pp. 285-286). For each path of the volatility, we can compute two path-conditional option bounds.

$$\bar{C}(S_t, x_t, x_{t+\Delta t}, \ldots, x_{T-\Delta T}) = e^{-r\Delta t} E^{U(x_t, x_{t+\Delta t}, \ldots, x_{T-\Delta T})}[(S_T - K)^+ \mid S_t, x_t],$$
$$\underline{C}(S_t, x_t, x_{t+\Delta t}, \ldots, x_{T-\Delta T}) = e^{-r\Delta t} E^{L(x_t, x_{t+\Delta t}, \ldots, x_{T-\Delta T})}[(S_T - K)^+ \mid S_t, x_t]$$

The assumption that the volatility is not priced enables us to compute the two option bounds as the means of the path conditional bounds.

$$\bar{C}(S_t, x_t) = E[\bar{C}(S_t, x_t, x_{t+\Delta t}, \ldots x_{T-\Delta T})],$$
$$\underline{C}(S_t, x_t) = E[\underline{C}(S_t, x_t, x_{t+\Delta t}, \ldots x_{T-\Delta T})]$$

I. Proof of Proposition 6

As argued in Wiggins (1987), the function $X(x,t)$ is increasing (decreasing) in x when $\alpha \geq 1$ ($\alpha \in (0,1]$). Since as shown in Merton (1973) $C(S_t, x_t)$ is an increasing function of x_t for any S_t, we have, from (6.8)

$$E[X(x_{t+\Delta t}, t+\Delta t)C(S_{t+\Delta t}, x_{t+\Delta t}) \left| x_t \right.] \geq (\leq) E[X(x_{t+\Delta t}, t+\Delta t) \left| x_t \right.] E[C(S_{t+\Delta t}, x_{t+\Delta t}) \left| x_t \right.],$$

whenever $\alpha \geq 1$ ($\alpha \in (0,1]$)
(A.13)

Replacing (A.13) into the RHS of (6.8) and using the independence of the error terms $\varepsilon_{t+\Delta t}$ and $\eta_{t+\Delta t}$ we get the following, for the case $\alpha \geq 1$ and for any $t < T - \Delta t$

$$C(S_t, x_t) \geq \frac{E[S_{t+\Delta t}^{-\alpha} E[C(S_{t+\Delta t}, x_{t+\Delta t}) | x_t] | S_t]}{E[S_{t+\Delta t}^{-\alpha} | S_t]}. \tag{A.14}$$

Set now $S_{t+\Delta t}^{-\alpha} \equiv Y_{t+\Delta t}$ in (6.7) and (A.14) and we have, noting that $Y_{t+\Delta t}$ is monotone decreasing in $S_{t+\Delta t}$ and the call option price is convex in $S_{t+\Delta t}$

$$C(S_t, x_t) \geq \frac{E[Y_{t+\Delta t} E[C(S_{t+\Delta t}, x_{t+\Delta t}) | x_t] | S_t]}{E[Y_{t+\Delta t} | S_t]},$$

$$\frac{E[Y_{t+\Delta t} S_{t+\Delta t} | S_t]}{E[Y_{t+\Delta t} | S_t]} = e^{r\Delta t} \tag{A.15}$$

Since at $t = T - \Delta t$ $\underline{C}(S_{T-\Delta t}, x_{T-\Delta t})$ as given by (6.4) is clearly a lower bound on $C(S_t, x_t)$, (A.15) shown clearly that the recursive expressions for the lower bound $\underline{C}(S_t, x_t)$ in (6.4) are also lower bounds on $C(S_t, x_t)$ here as well. The proof for the case $\alpha \in (0,1]$ is symmetrical, QED.

J. Proof of Proposition 7

For any t the probability distribution of the payoff $(S_T - K)^+$ conditional on (S_t, h_{t+1}) is independent of S_t for any given h_{t+1}. Hence, by Merton's Theorem 10 we know that $C(S_t, h_{t+1})$ is convex in S_t for any given h_{t+1}. We prove the Proposition by using induction to prove the following: $\bar{C}(S_t, h_{t+1})$ and $\underline{C}(S_t, h_{t+1})$ are convex in S_t for any given h_{t+1} and are bounds on admissible values of $C(S_t, h_{t+1})$.

At T-1 the above statement is obviously true. Assume that at t+1 $\bar{C}(S_{t+1}, h_{t+2})$ and $\underline{C}(S_{t+1}, h_{t+2})$ are convex in S_{t+1} for any given h_{t+2} and contain admissible values of $C(S_{t+1}, h_{t+2})$. Consider the expressions for $\bar{C}(S_t, h_{t+1})$ and $\underline{C}(S_t, h_{t+1})$ as given by (7.3). These expressions are convex, since the distribution of the return $\frac{S_{t+1}}{S_t}$ is predictable given h_{t+1}. Consider now problem (3.4) and replace $C(S_{t+1}, h_{t+2})$ by $\bar{C}(S_{t+1}, h_{t+2})$ and $\underline{C}(S_{t+1}, h_{t+2})$ in the maximization and minimization problems respectively. Then the problem (3.4) yields the (not necessarily convex) solutions $\tilde{C}(S_{t+1}, h_{t+2} \mid S_t, h_{t+1})$ and $\underset{\sim}{C}(S_{t+1}, h_{t+2} \mid S_t, h_{t+1})$ at t. If $Z(h_{t+1})$ denotes the returns $\frac{S_{t+1}}{S_t}$, which take, without loss of generality, n discrete values z_i $i = 1, \ldots, n$ then we know from Ritchken (1985) that $\tilde{C}(S_{t+1}, h_{t+2} \mid S_t, h_{t+1})$ and $\underset{\sim}{C}(S_{t+1}, h_{t+2} \mid S_t, h_{t+1})$ lie on the convex hull of the n expressions $E[C(S_{t+1}, h_{t+2}) | Z(h_{t+1}) \leq z_i, S_t, h_{t+1}]$, graphed against the conditional means $E[Z(h_{t+1}) | Z(h_{t+1}) \leq z_i, S_t, h_{t+1}]$, $i = 1, \ldots, n$. These clearly contain the bounds given by (7.3), QED.