Jump-Diffusion Option Valuation Without a Representative Investor: a Stochastic Dominance Approach

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This version February 2010

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Abstract

We present a new method of pricing plain vanilla call and put options when the underlying asset returns follow a jump-diffusion process. The method is based on stochastic dominance insofar as it does not need any assumption on the utility function of a representative investor apart from risk aversion. It develops discrete time multiperiod reservation write and reservation purchase bounds on option prices. The bounds are valid for any asset dynamics and are such that any risk averse investor improves her expected utility by introducing a short (long) option in her portfolio if the upper (lower) bound is violated by the observed market price. The bounds are evaluated recursively and the limiting forms of the bounds are found as time becomes continuous. It is found that the two bounds tend to the common limit equal to the Black-Scholes-Merton (BSM) price when there is no jump component, but to two different limits when the jump component is present.

Keywords: option pricing; incomplete markets; stochastic dominance; jump-diffusion

I. Introduction

The presence of extreme events or "jumps" in the probability distribution of stock returns is one of the earliest examples in the financial literature of a dynamically incomplete market. As Merton (1976) was the first to point out, in such markets arbitrage methods are unable by themselves to price contingent claims without additional assumptions about financial market equilibrium. In modeling such equilibrium the extreme events, which are by definition rare, are priced by introducing a representative investor whose portfolio decisions define the market's pricing mechanism. The resulting option prices are almost always functions of the investor's attitudes towards risk, as well as of the parameters of the return distributions of the underlying asset.

In this paper we introduce a different approach to contingent claims pricing in the presence of rare event risk. We derive option prices that are functions of the parameters of the underlying asset return distribution and do not depend on the attitude towards risk of a representative investor. Since the financial markets are incomplete the prices that we derive are not unique but lie in an interval whose width is a function of the risk premium of the underlying asset;[3] except for this parameter, the prices that we derive use the same information set as the existing approaches. We first derive results for index options and then extend them to individual stock options. Our numerical results indicate that for "reasonable" values of the parameters of the return process the width of the interval does not exceed 8% for at-the-money S&P 500 index options.

The main motivation for our approach is the fact that investor attitudes towards risk are not directly observable and their estimates are notoriously unstable and unreliable. Estimates extracted from empirical research show a complete disconnect between option-based models and studies that model directly the consumption of "representative" investors, with the latter estimates being as much as ten times as large as the former![4] We discuss in the next section the effects of these large estimates on the traditional approach to option pricing by the representative investor. On the other hand we note that the realized return of the underlying asset is certainly observable and that there are statistical methods to extract efficient estimates of the parameters of the actual distribution of the returns of that asset, which in many empirical studies is assumed to be a mixture of a diffusion process (possibly with stochastic volatility), and rare events following Poisson arrivals with log-normally distributed jump amplitudes.[5]

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[3] While the risk premium is itself determined in equilibrium by the attitudes towards risk of the investors in the market, we adopt very weak assumptions for this equilibrium, such as not requiring the existence of a representative investor or particular types of utility functions.

[4] Bates (1991) uses a value of 2 for his option pricing results, while econometric studies in option markets by Engle and Rosenberg (2002) and Bliss and Panigirtzoglou (2004) report findings of 2 to12 and 1.97 to 7.91 respectively. On the other hand Kandell and Stambaugh (1991) give estimates as high as 30, while the equity premium puzzle literature reports empirical findings of 41 in Mehra and Prescott (1985) and more than 35 in Campbell and Conchrane (1999). See also the survey article by Kotcherlakota (1996).

[5] There is a large econometric literature on the estimation of the parameters of such mixed processes from historical data. See, in particular, Ait-Sahalia (2004), Bates (2000), Eraker *et al* (2003), and Tauchen and Zhou (2007). Note that our results do not depend on the assumption of lognormal jump amplitude distributions.

Our results are developed in a multiperiod discrete time formulation of market equilibrium in an incomplete markets setup and for any type of asset dynamics. We assume that there exists a certain class of risk averse traders in the financial markets who hold only the underlying asset, the riskless asset and (possibly) the option.[6] For such traders we derive stochastic dominance reservation write and reservation purchase option prices, implying that any risk averse trader would improve her utility function by introducing the corresponding short or long option in her portfolio if the market price exceeds the corresponding bound. Since market equilibrium is incompatible with the existence of such dominant strategies, it follows that in equilibrium the observed option prices should lie between the derived bounds.

For our assumed class of traders the market equilibrium conditions imply that the pricing kernel in the discrete time setup is monotone decreasing with respect to the underlying asset return within a single trading period, but not necessarily when there are more than one periods to option expiration. From this property we extract two boundary risk neutral distributions under which the expected option payoffs evaluated recursively for any type of periods to expiration define the region of admissible option prices. We also introduce a suitable discretization of the return process that tends to a desired continuous time process at the limit of continuous trading. The interval of option values in continuous time under jump-diffusion asset dynamics is then found by considering the continuous time limits of these expected payoffs under the two boundary distributions. We also show that the interval shrinks to a single value, the Black-Scholes-Merton (BSM, 1973) option pricing model, when the jump component of the return is set equal to zero.

We note that the assumption about the monotonicity of the pricing kernel in a single trading period, the only property that is necessary for the derivation of our bounds, is also satisfied in *all the models* of contingent claims pricing under jump-diffusion processes that have appeared in the literature so far. In the equilibrium models that use a representative investor with a CPRA utility this monotonicity is implied by the joint dynamics of the stock return and the wealth of the representative investor, as well as by the shape of the indirect utility function.[7] These features also preserve the monotonicity in equilibrium models with more elaborate assumptions that include behavioral considerations such as uncertainty aversion and recursive utility.[8] Unlike stochastic dominance, these approaches assume specific expressions for the pricing kernel.

Our results are closely related to earlier studies of option pricing in incomplete markets in a discrete time framework. These were introduced by Perrakis and Ryan (1984), with important extensions by Ritchken (1985), Levy (1985), Perrakis (1986, 1988) and Ritchken and Kuo (1988). Constantinides and Perrakis (2002, 2007) introduced proportional transaction costs into these models and showed that they were capable of producing useful results under such conditions, unlike arbitrage- or equilibrium-based models. Empirical applications of this discrete time approach to the pricing of S&P 500

[6] This assumption is eventually relaxed to allow the derivation of stock options.
[7] See, for instance, Bates (1991), Amin and Ng (1993) and Amin (1993).
[8] See Liu, Pan and Wang (2005).

index options were presented in Constantinides, Jackwerth and Perrakis (2009). This paper provides a linkage between these discrete time models and the continuous time framework in which most option pricing work was done.

The paper is organized as follows, with all proofs consigned to the appendix. In the next section we present the discrete time representation of the stock returns and introduce the Lindeberg condition, the weak convergence criterion to continuous time processes. We show that under this criterion our representation converges to a generalized version of the mixed jump-diffusion process in continuous time, with possibly time- or state-dependent parameters and independent Poisson arrivals of the jump components with any type of distribution for their jump amplitudes. In section 3 we present the discrete time stochastic dominance option pricing results and apply them to our representation of the stock returns. We introduce the discrete time financial market equilibrium based on the monotonicity of the pricing kernel under general conditions about the utility function of a risk averse investor and for an unspecified return distribution. We then derive the two martingale probability distributions, transforms of the physical return distribution, that define the upper and lower bounds in a multiperiod context for a distribution with time-dependent returns within which the admissible option prices lie. These transforms are applied in section 4 to the discrete time representation of our return process introduced in section 2 in order to derive the two martingale probabilities defining the upper and lower bounds of admissible option prices in discrete time. We then apply the convergence criterion presented in section 2 to these two boundary risk neutral distributions, first for the simple diffusion case and then for jump-diffusion. We show that in the diffusion case the two bounds tend to the *same* limit, the BSM model, thus establishing the discrete time option bounds as generalizations of the Cox-Ross-Rubinstein (1979) binomial model. For jump-diffusion we show that the option bounds arise in closed form as solutions to two distinct partial differential equations. Section 5 extends these results to stock options by allowing the traders to hold marginal positions in individual stocks and their options, while the last section discusses the extension of the results to returns combining stochastic volatility with jumps and concludes.

In the remainder of this section we complete the literature review. Jump-diffusion processes were first introduced into option pricing models by Merton (1976), who derived a unique option price by assuming that the rare event risk was fully diversifiable and thus not priced. This assumption was clearly untenable following the 1987 stock market crash, and the next generation of option pricing models reflected attempts to relax it. Bates, (1988, 1991) introduced a general equilibrium model with a representative investor with CPRA utility, a modified version of Cox-Ingersoll-Ross (CIR, 1985) that incorporated correlated jump-diffusion processes in both the underlying asset and the investor wealth. Essentially the same model was also used by Amin and Ng (1993) and Amin (1993).

A standard feature of these valuation models was the dependence of the derived option prices on several parameters over and above the ones defining the jump-diffusion return distribution: the mean and variance of the log-amplitudes of the jump components in the investor wealth distribution; the covariance of these same amplitudes with the corresponding measures in the return distribution; the risk aversion parameter of the representative investor.[9] These parameters modify the jump intensity and expected log-amplitude parameters of the physical return process. In the more general model by Liu, Pan and Wang (2005) of option pricing under jump-diffusion that includes uncertainty aversion the martingale probability used in valuing contingent claims is also modified by the uncertainty aversion parameters of the utility function of the representative investor.[10]

In empirical tests the jump-diffusion model is often included in a nested model that also includes stochastic volatility. Many of these tests were motivated by the need to explain the well-known volatility smile originally documented by Rubinstein (1994); only a few will be mentioned here. Bates (1996) applied the nested models to Deutsche mark currency options, Bates (2000) to S&P 500 futures options, Pan (2002) and Rosenberg and Engle (2002) to S&P 500 index options, and Bliss and Panigirtzoglou (2004) to FTSE 100 and S&P 500 index futures options. In those tests the parameters of the implied risk neutral distribution are extracted from cross sections of observed option prices and attempts are made to reconcile these option-based distributions with data from the market of the underlying asset. All studies stress the importance of jump risk premia in these reconciliation attempts. The results of this paper, by introducing an additional set of requirements in the valuation of options under jump-diffusion have obvious implications for future empirical work.[11]

II. <u>The Stock Return Model</u>

We consider a market with an underlying asset (the stock) with current price S_t and a riskless asset with return *per period* equal to R. There is also a European call option with strike price K expiring at some future time T. Time is initially assumed discrete $t = 0,1,...,T$, with intervals of length Δt, implying that $R = e^{r\Delta t} = 1 + r\Delta t + o(\Delta t)$, where r denotes the interest rate in continuous time. In each interval the underlying asset has returns $\dfrac{S_{t+\Delta t} - S_t}{S_t} \equiv z_{t+\Delta t}$, whose distribution may depend on S_t.[12]

[9] See, for instance, equation (10) in Bates (1991), equation (27) of Amin and Ng (1993), or equation (33) of Amin (1993). If the models are applied to index options with the index considered a proxy for investor wealth then the need for separate estimates of wealth distribution parameters disappears, but the risk aversion parameter remains in the expressions.

[10] See their equations (19)-(21).

[11] As Eraker *et al* (2003, p. 1294) point out, the joint estimation of parameters from both stock and option data does not necessarily reduce the uncertainty in the estimates, unless the jump risk premia are arbitrarily restricted. Our results provide objective methods of restricting such premia and, thus, improve the estimates.

[12] More complex dependence of the returns on other underlying state variables like, for instance, stochastic volatility will be discussed in subsequent sections.

We model the returns as a sum of two components, one of which will tend to diffusion and the other to a jump process. With probability $1-\lambda_t \Delta t$ the return has the following form[13]

$$z_{t+\Delta t} = [\mu(S_t, t) - \lambda_t \mu_J]\Delta t + \sigma(S_t, t)\varepsilon\sqrt{\Delta t}. \tag{2.1}$$

In this expression ε has a bounded distribution of mean zero and variance one, $\varepsilon \sim D(0,1)$ and $0 < \varepsilon_{\min} \le \varepsilon \le \varepsilon_{\max}$, but otherwise unrestricted.

With probability $\lambda_t \Delta t$ there is a jump J_t in the return. The jump is a random variable with distribution D_{Jt} with mean μ_{Jt} and variance σ_{Jt}. In most of the literature it is assumed that $J_t = J$ and D_J is lognormal, that is $\ln(1+J)$ has a normal distribution. Although our results may be extended to allow for dependence of both jump intensity and jump amplitude distribution on S_t, we shall adopt the assumption that the jump process is state- and time-independent, with $\lambda_t = \lambda,\ J_t = J$. On the other hand the distribution D_J is restricted only by the requirement that the return cannot be less than -1, although we assume, without loss of generality, that the variable J takes both positive and negative values with $-1 \le J_{\min} < 0 < J_{\max}$. With this specification the return becomes, if we set $\mu(S_t, t) \equiv \mu_t,\ \sigma(S_t, t) \equiv \sigma_t$

$$z_{t+\Delta t} = (\mu_t - \lambda\mu_J)\Delta t + \sigma_t \varepsilon\sqrt{\Delta t} + J\Delta N\ . \tag{2.2}$$

where N is a Poisson counting process with intensity λ. An alternative representation of the discretization (2.2), which will also be used in the proofs, is the description of its outcomes

$$z_{t+\Delta t} = \begin{cases} (\mu_t - \lambda\mu_J)\Delta t + \sigma_t\varepsilon\sqrt{\Delta t} & \text{with probability} \quad 1-\lambda\Delta t \\ (\mu_t - \lambda\mu_J)\Delta t + \sigma_t\varepsilon\sqrt{\Delta t} + J & \text{with probability} \quad \lambda\Delta t \end{cases}, \tag{2.2a}$$

The "traditional" approach to the pricing of index options when the index returns are given by (2.2) or (2.2a) is to value the options as the discounted expectations of their payoffs under a risk-adjusted distribution of the form $z_{t+\Delta t} = (r - \lambda^* \mu_J^*)\Delta t + \sigma_t\varepsilon\sqrt{\Delta t} + J^*\Delta N$, where both λ^* and μ_J^* have been distorted by the risk aversion parameter γ of a CPRA utility function; see Bates (1991, p. 1034). In particular $\lambda^* = \lambda\exp(-\gamma\mu_J + \frac{1}{2}\gamma(1+\gamma)\sigma_J^2)$, which is approximately equal to λ if γ and σ_J^2 are small. For the estimated parameter values of the jump-diffusion estimates for the S&P 500 of the recent study of Tauchen and Zhou (2007) of

[13] For simplicity dividends are ignored throughout this paper. All results can be easily extended to the case where the stock has a known and constant dividend yield, as in index options. In the latter case the instantaneous mean in (2.1) and (2.2) is net of the dividend yield.

$\lambda = 0.13,\ \sigma_J = 0.54,\ \mu_J = 0.05$ the parameter λ^* becomes very large for all but the smallest risk aversion estimates mentioned in footnote 4 of the previous section and corresponds to unreasonably large option values.

In this section we present the conditions that establish the convergence of the processes described by (2.1) and (2.2) respectively to diffusion and to a mixed jump-diffusion process. In the next section we discuss the market equilibrium and derive the discrete time bounds on admissible option values supported by such equilibrium. The convergence criteria presented here are then applied to the processes under which these bounds are derived in order to find the option values under continuous time diffusion and mixed processes.

To prove the convergence of option prices, we rely on the weak convergence of the underlying price process, first to a diffusion and then to a jump diffusion. For any number m of time periods to expiration we define a sequence of stock prices $\{S_t \mid \Delta t, m\}$ and an associated probability measure P^m. The weak convergence property for such processes[14] stipulates that for any continuous bounded function f we must have $E^{P_m}\left[f(S_T^m)\right] \to E^P\left[f(S_T)\right]$, where the measure P corresponds to diffusion limit of the process, to be defined shortly. P_m is then said to converge weakly to P and S_T^m is said to converge in distribution to S_T. A necessary and sufficient condition for the convergence to a diffusion is the Lindeberg condition, which was used by Merton (1992) to develop criteria for the convergence of multinomial processes. In a general form, if ϕ_t denotes a discrete stochastic process in d-dimensional space the Lindeberg condition states that a necessary and sufficient condition that ϕ_t converges weakly to a diffusion, is that for any fixed $\delta > 0$ we must have

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\|\phi - \varphi\| > \delta} Q_{\Delta t}(\phi, d\varphi) = 0 \tag{2.3}$$

where $Q_{\Delta t}(\phi, d\varphi)$ is the transition probability from $\phi_t = \phi$ to $\phi_{t+\Delta t} = \varphi$ during the time interval Δt. Intuitively, it requires that ϕ_t does not change very much when the time interval Δt goes to zero.

When the Lindeberg condition is satisfied, the following limits exist

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\|\varphi - \phi\| < \delta} (\varphi_i - \phi_i) Q_{\Delta t}(\phi, d\varphi) = \mu_i(\phi) \tag{2.4}$$

[14] For more on weak convergence for Markov processes see Ethier and Kurz (1986), or Strook and Varadhan (1979).

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{\|\varphi - \phi\| < \delta} (\varphi_i - \phi_i)(\varphi_j - \phi_j) Q_{\Delta t}(\phi, d\varphi) = \sigma_{ij}(\phi) \tag{2.5}$$

The conditions (2.3), (2.4) and (2.5) are equivalent to the weak convergence of the discrete process to a diffusion process with the generator[15]

$$\mathcal{A} = \frac{1}{2} \sum_{i=1, j=1}^{d} \sigma_{ij} \frac{\partial^2}{\partial \phi_i \partial \phi_j} + \sum_{i=1}^{d} \mu_i \frac{\partial}{\partial x_i} \,. \tag{2.6}$$

By the definition of the generator, for each bounded, real valued function f we have

$$\lim_{\Delta t \to 0} \frac{Ef(\phi_{t+\Delta t}) - f(\phi_t)}{\Delta t} = \mathcal{A} f \tag{2.7}$$

In our case the state variable vector is one-dimensional and $\phi_t = S_t$. With these definitions we have the following result, proven in the appendix.

Lemma 1. *For $\lambda = 0$ the discrete process described by equation (2.1) converges weakly to the following diffusion (2.8), where W is a Wiener process with $E(dW) = 0$ and $Var(dW) = dt$*

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW \,. \tag{2.8}$$

It can be easily seen that the process given by (2.2) does not satisfy the Lindeberg condition, since

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} Q_t(\delta) = \lambda \int_{|z_t| > \delta} dD_J(J) + \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{|z_t| > \delta} dD(\varepsilon)$$

As shown in the proof of Lemma 1 for the diffusion case, the second integrand is zero for Δt sufficiently low. However, the first integrand is strictly positive for any Δt, implying that the process does not converge to diffusion in continuous time. The limit result for (2.2) is given by the following lemma, proven in the appendix.

Lemma 2. *The discrete process described by (2.2) converges weakly to the jump-diffusion process (2.9):*

$$\frac{dS_t}{S_t} = (\mu_t - \lambda \mu_J) dt + \sigma_t dW + J dN \,. \tag{2.9}$$

Having established that (2.2) and (2.1) are valid discrete time representations of a mixed

[15] See for instance Merton (1992) for a discussion on the generators of diffusions and jump processes.

process and its diffusion component, we now turn to contingent claims pricing for such processes.

III. The Option Pricing Model in Discrete Time

Except for the trivial case where the random variable ε takes only two values the market for the stock with returns given by (2.1) is incomplete in a discrete time context. The same is of course true *a fortiori* for the stock with returns given by (2.2). The valuation of an option in such a market cannot yield a unique price. Our market equilibrium is derived under the following set of assumptions that are sufficient for our results:

> There exists at least one utility-maximizing risk averse investor (the *trader*) in the economy who holds only the stock and the riskless asset[16]
>
> This particular investor is marginal in the option market
>
> The riskless rate is non-random[17]

Each trader holds a portfolio of x_t in the riskless asset and y_t in the stock by maximizing recursively the expected utility of final wealth[18] over the periods $t = 0,1,...,T'$ of length Δt. The current value function is $\Omega(x_t + y_t \left| S_t \right.) = Max_{v_t} E[\Omega((x_t - v_t)R + (y_t + v_t)(1 + z_{t+\Delta t}) \left| S_t \right.]$, where v_t denotes the optimal portfolio revision or stock purchase from the riskless account. If the trader also has a marginal open position in a given call option with value $C_t(S_t)$ and with terminal condition $C_T = (S_T - K)^+$ at option expiration time $T < T'$ [19] then the following relations characterize market equilibrium in any *single* trading period $(t, t+\Delta t)$, assuming no transaction costs and no taxes:

$$\begin{aligned} &E[Y(z_{t+\Delta t}) \left| S_t \right.] = R^{-1}, \ \ E[(1+z_{t+\Delta t})Y(z_{t+\Delta t}) \left| S_t \right.] = 1, \\ &C_t(S_t) = E[C_{t+\Delta t}(S_t(1+z_{t+\Delta t}))Y(z_{t+\Delta t}) \left| S_t \right.] \end{aligned} \quad , \qquad (3.1)$$

In (3.1) $Y(z_{t+\Delta t})$ represents the pricing kernel, the state-contingent discount factor or normalized marginal rate of substitution of the trader evaluated at her optimal portfolio choice. Because of the assumed risk aversion and portfolio composition of our traders it can be easily seen that the pricing kernel would possess the following property:

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[16] This assumption is relaxed in Section V.

[17] Although this assumption may not be justified in practice, its effect on option values is generally recognized as minor in short- and medium-lived options. It has been adopted without any exception in *all* equilibrium based jump-diffusion option valuation models that have appeared in the literature. See the comments in Bates (1991, p. 1039, note 30) and Amin and Ng (1993, p. 891). In order to evaluate the various features of option pricing models, Bakshi, Cao and Chen (1997) applied without deriving it a risk-neutral model featuring stochastic interest rate, stochastic volatility and jumps. They found that stochastic interest rates offer no goodness of fit improvement.

[18] The results are unchanged if the traders are assumed to maximize the expected utility of the consumption stream.

[19] All results in this paper are derived for call options. They are applicable without reformulation to European put options, either directly or through put-call parity.

Property: *The pricing kernel* $Y(z_{t+\Delta t})$ *is monotone (either non-increasing or non-decreasing) in the stock return* $z_{t+\Delta t}$ *for every* $t = 0,1,...,T$.

This property is sufficient for the derivation of tight option bounds for all stock return distributions and not only those given by (2.2). It can be easily seen from the second relation in (3.1) that under such an assumption $Y(z_{t+\Delta t})$ must be non-increasing if the optioned stock is a "positive beta" one, with expected return exceeding the riskless rate, since this implies that the trader will always hold a positive amount of the stock. Since this is the case for the overwhelming majority of stocks, this is the assumption that will be adopted here. The same set of assumptions also underlies the stochastic dominance option bounds of Constantinides and Perrakis (2002, 2007) that were derived under general distributional assumptions and included proportional transaction costs. These assumptions may be restrictive for options on individual stocks, but their validity in the case of index options cannot be doubted, given that fact that numerous surveys have shown that a large number of US investors follow indexing strategies in their investments.[20] These market equilibrium assumptions are quite general, insofar as they allow the existence of other investors with different portfolio holdings than the trader. They do not assume the existence of a representative investor, let alone one with a specific type of utility function. The results presented in this section are derived for unspecified discrete time asset dynamics, and are applied to the specific case of jump-diffusion in the next section.

A more restrictive set of market equilibrium assumptions underlies the well-known jump-diffusion option valuation models of Bates (1988, 1991) and Amin and Ng (1993), and their more recent extension by Liu, Pan and Wang (2005) that include behavioral considerations. In the case of index options these studies assume that there is a representative consumer with a time-additive CPRA utility function of consumption over a finite or an infinite horizon.[21] In all those models the indirect utility is a concave function of investor wealth, implying that the marginal utility is decreasing. If, as is commonly the case, the correlations between both diffusion and jump components of the stock and the wealth processes are positive then the conditional expectation of the marginal utility given $z_{t+\Delta t}$ is decreasing, implying clearly that $Y(z_{t+\Delta t})$ is non-increasing. Thus, the bounds derived by our market equilibrium assumptions are also applicable to these earlier models as well.

[20] Bogle (2005) reports that in 2004 index funds accounted for about one third of equity fund cash inflows since 2000 and represented about one seventh of equity fund assets.

[21] In the case of stock options the market equilibrium assumptions must also include the following: both stock and aggregate wealth (or consumption) are jump-diffusion processes with state- and time-independent jump component; (one plus) the percentage jump amplitudes in both wealth and stock follow a bivariate lognormal distribution.

The derivation of option pricing bounds under a non-increasing pricing kernel $Y(z_{t+\Delta t})$ can be done with at least two different approaches, the expected utility comparisons under a zero-net-cost option strategy introduced by Perrakis and Ryan (1984) and the linear programming (LP) method pioneered by Ritchken (1985). In the former approach an option upper bound is found by having the trader open a short position in an option with price C, with the amounts αC and $(1-\alpha)C$ added respectively to the riskless asset and the stock account. For the option lower bound a long position is financed by shorting an amount βS_t, β<1 of stock, with the remainder invested in the riskless asset. Both bounds are found as limits on the call price C such that the value function of the investor with the open option position would exceed that of the trader who does not trade in the option if the write (purchase) price of the call lies above (below) the upper (lower) limit on C. This approach[22] yields results that are *identical* to the ones of the LP approach, which is the one presented here.

The distribution of the return $z_{t+\Delta t}$ is assumed discrete, with the continuous case arising as the obvious limit as the number of states becomes progressively denser. We denote this distribution by $P(z_{t+\Delta t}|S_t)$, which may depend on S_t in the most general case; for notational simplicity this dependence is suppressed in the expressions that follow. For the discrete distribution case at any time t the stock's return is z_{tj} in state j, where j is an index, such that $z_{1t} \le z_{2t} ... \le z_{nt}$ .The probabilities of the n states are $p_{1t}, p_{2t}, ..., p_{nt}$. The pricing kernel, the state-contingent discount factors, are denoted by $Y_{1t}, \ldots, Y_{nt}$, and it is assumed that $Y_{1t} \ge Y_{2t} \ge ... \ge Y_{nt}$. Let also $\overline{C}_t(S_t)$ and $\underline{C}_t(S_t)$ denote respectively the upper and lower bounds on admissible call option prices supported by the market equilibrium (3.1), the asset dynamics and the monotonicity of the pricing kernel assumption.

If the option price function $C(S_{t+\Delta t}) = C(S_t(1+z_{t+\Delta t}))$ is known then bounds on $C(S_t)$ are found by solving the following LP, for given distributions $(z_{jt+\Delta t}, p_{jt+\Delta t})$. This LP evaluates the reservation write and reservation purchase prices of the option under market equilibrium that excludes the presence of stochastically dominant strategies, namely strategies that augment the expected utility of all traders. Violations of the bounds given by the LP imply that *any* such trader can improve her utility by introducing a corresponding short or long option in her portfolio.

[22] It is presented in an appendix, available from the authors on request.

$$\max_{Y_{jt+\Delta t}} \sum_{j=1}^{j=n} C_{t+\Delta t}(S_t(1+z_{jt+\Delta t}))p_{jt+\Delta t}Y_{jt+\Delta t}$$

$$(\min_{Y_{jt+\Delta t}} \sum_{j=1}^{j=n} C_{t+\Delta t}(S_t(1+z_{jt+\Delta t}))p_{jt+\Delta t}Y_{jt+\Delta t})$$

$$\text{subject to:} \qquad (3.2)$$

$$1 = \sum_{j=1}^{n}(1+z_{jt+\Delta t})p_{jt+\Delta t}Y_{jt+\Delta t}$$

$$R^{-1} = \sum_{j=1}^{n} p_{jt+\Delta t}Y_{jt+\Delta t}$$

$$Y_{1t+\Delta t} \geq Y_{2t+\Delta t} \geq \ldots \geq Y_{nt+\Delta t}$$

Define also the following conditional expectations:

$$\hat{z}_{jt+\Delta t} = \frac{\sum_{i=1}^{j} z_{it+\Delta t} p_{it+\Delta t}}{\sum_{i=1}^{j} p_{it+\Delta t}} = E\left[z_{t+\Delta t} \mid z_{t+\Delta t} \leq z_{jt+\Delta t}, S_t\right], \quad j = 1,\ldots,n \qquad (3.3)$$

With these definitions it is clear that $\hat{z}_{nt+\Delta t} = E[z_{t+\Delta t} | S_t] \equiv \hat{z}$ and by assumption $1+\hat{z}_{nt+\Delta t} \geq R$.[23] Similarly we have $\hat{z}_{1t+\Delta t} = z_{1t+\Delta t} = z_{\min,t+\Delta t}$, the lowest possible return, which will be initially assumed strictly greater than -1.

The constraints in the LP differ from the general market equilibrium relations (3.1) by the last set of inequalities in (3.2) that correspond to the monotonicity of the pricing kernel assumption. In the absence of this inequality set it can be shown that the results of the LP yield the well-known no-arbitrage bounds derived by Merton (1973),[24] the only bounds on admissible option prices that rely only on absence of arbitrage and on no other assumption about the market equilibrium process. The following important result, proven in the appendix, characterizes the solution of the LP in (3.2):

<u>Lemma 3</u>: *If the option price $C_t(S_t)$ is convex for any t then it lies within the following bounds*

---

[23] Similar expressions as the ones presented in Lemma 3 and Proposition 1 also hold when we have a "negative beta" stock, with $Y(z_{t+\Delta t})$ is non-decreasing and $1+\hat{z}_{nt+\Delta t} < R$. This case is presented in an appendix, available from the authors on request. The limiting results of the next section also hold for this case as well, with minor modifications.

[24] See Ritchken (1985, section III). Actually, the upper bound in that LP is equal to the stock price minus the strike price discounted by the highest possible return; this last term goes to 0 in the multiperiod case.

$$\frac{1}{R}E^{L_t}[C_{t+\Delta t}(S_t(1+z_{t+\Delta t}))] \le C_t(S_t) \le \frac{1}{R}E^{U_t}C_{t+\Delta t}(S_t(1+z_{t+\Delta t})), \tag{3.4}$$

where E^{U_t} *and* E^{L_t} *denote respectively expectations taken with respect to the distributions*

$$\begin{aligned}
U_{1t} &= \frac{R-1-\hat{z}_{1t+\Delta t}}{\hat{z}-\hat{z}_{1t+\Delta t}}p_{1t+\Delta t} + \frac{\hat{z}+1-R}{\hat{z}-\hat{z}_{1t+\Delta t}} \\
U_{jt} &= \frac{R-1-\hat{z}_{1t+\Delta t}}{\hat{z}-\hat{z}_{1t+\Delta t}}p_{jt+\Delta t},\ j=2,\ldots,n
\end{aligned}, \tag{3.5a}$$

$$\begin{aligned}
L_{jt} &= \frac{\hat{z}_{h+1,t+\Delta t}+1-R}{\hat{z}_{h+1,t+\Delta t}-\hat{z}_{ht+\Delta t}}\frac{p_{jt+\Delta t}}{\sum_{k=1}^{h}p_{kt+\Delta t}} + \frac{R-1-\hat{z}_{ht+\Delta t}}{\hat{z}_{h+1,t+\Delta t}-\hat{z}_{ht+\Delta t}}\frac{p_{jt+\Delta t}}{\sum_{k=1}^{h+1}p_{kt+\Delta t}},\ j=1,\ldots,h \\
L_{h+1,t} &= \frac{R-1-\hat{z}_{ht+\Delta t}}{\hat{z}_{h+1,t+\Delta t}-\hat{z}_{ht+\Delta t}}\frac{p_{h+1,t+\Delta t}}{\sum_{k=1}^{h+1}p_{kt+\Delta t}},\ L_{jt}=0,\ j>h+1
\end{aligned} \tag{3.5b}$$

In the expressions (3.5b) h *is a state index such that* $\hat{z}_{ht+\Delta t} \le R-1 < \hat{z}_{h+1,t+\Delta t}$.

For a continuous distribution $P(z_{t+\Delta t}|S_t)$ *of the stock return the expectations are taken with respect to the following distributions*

$$\begin{aligned}
U(z_{t+\Delta t}) &= \begin{cases} P(z_{t+\Delta t}|S_t) & \text{with probability} \quad \frac{R-1-z_{\min,t+\Delta t}}{E(z_{t+\Delta t})-z_{\min,t+\Delta t}} \\ 1_{z_{\min,t+\Delta t}} & \text{with probability} \quad \frac{E(z_{t+\Delta t})+1-R}{E(z_{t+\Delta t})-z_{\min,t+\Delta t}} \equiv Q \end{cases}, \\
L(z_{t+\Delta t}) &= P(z_{t+\Delta t} \mid S_t, z_{t+\Delta t} \le z_t^*),\ E(1+z_{t+\Delta t}|S_t, z_{t+\Delta t} \le z_t^*) = R
\end{aligned} \tag{3.6}$$

With this LP it can be shown that the bounds $\overline{C}_t(S_t)$ and $\underline{C}_t(S_t)$ may be derived recursively by a procedure described in Proposition 1. This procedure yields a closed form solution, which relies heavily on the assumed convexity of the option price $C_t(S_t)$, itself a consequence of the convexity of the payoff. The convexity property clearly holds for the diffusion and jump-diffusion cases examined in this paper.[25]

Proposition 1*: Under the monotonicity of the pricing kernel assumption and for a discrete distribution of the stock return* z_t *all admissible option prices lie between the upper and lower bounds* $\overline{C}_t(S_t)$ *and* $\underline{C}_t(S_t)$, *evaluated by the following recursive*

[25] The convexity of the option with respect to the underlying stock price holds in all cases in which the return distribution had iid time increments, in all univariate state-dependent diffusion processes, and in bivariate (stochastic volatility) diffusions under most assumed conditions; see Merton (1973) and Bergman, Grundy and Wiener (1996).

expressions

$$
\begin{aligned}
&\overline{C}_T(S_T) = \underline{C}_T(S_T) = (S_T - K)^+ \\
&\overline{C}_t(S_t) = \frac{1}{R} E^{U_t}[\overline{C}_{t+\Delta t}(S_t(1+z_{t+\Delta t}))\,|\,S_t] \qquad , \\
&\underline{C}_t(S_t) = \frac{1}{R} E^{L_t}[\underline{C}_{t+\Delta t}(S_t(1+z_{t+\Delta t}))\,|\,S_t]
\end{aligned} \tag{3.7}
$$

where E^{U_t} *and* E^{L_t} *denote expectations taken with respect to the distributions given in (3.5ab) or (3.6).*

Proof: We use induction to prove that (3.7) yields expressions that form upper and lower bounds on admissible option values. It is clear that (3.7) holds at *T* and that $\overline{C}_T(S_T)$ and $\underline{C}_T(S_T)$ are both convex in S_T. Assume now that $\overline{C}_{t+\Delta t}(S_t(1+z_{t+\Delta t}))$ and $\underline{C}_{t+\Delta t}(S_t(1+z_{t+\Delta t}))$ are respectively upper and lower bounds on the convex function $C_{t+\Delta t}(S_t(1+z_t))$, implying that

$$\frac{1}{R}\underline{C}_{t+\Delta t}(S_t(1+z_{t+\Delta t})) \le C_{t+\Delta t}(S_t(1+z_{t+\Delta t})) \le \frac{1}{R}\overline{C}_{t+\Delta t}(S_t(1+z_{t+\Delta t})) \tag{3.8}$$

By Lemma 3 we also have

$$\frac{1}{R}E^{L_t}[C_{t+\Delta t}(S_t(1+z_{t+\Delta t}))] \le C_t(S_t) \le \frac{1}{R}E^{U_t}C_{t+\Delta t}(S_t(1+z_{t+\Delta t})) \tag{3.9}$$

(3.8) and (3.9), however, imply that

$$\underline{C}(S_t) = \frac{1}{R}E^{L_t}[\underline{C}_{t+\Delta t}(S_t(1+z_{t+\Delta t}))] \le C_t(S_t) \le \frac{1}{R}E^{U_t}\overline{C}_{t+\Delta t}(S_t(1+z_{t+\Delta t})) = \overline{C}(S_t)\,, \tag{3.10}$$

QED.

An important special case arises when $z_{1t+\Delta t} = z_{\min,t+\Delta t} = -1$, implying that the stock can become worthless within a single elementary time period $(t, t+\Delta t)$. In such a case the lower bound given by expectations taken with (3.5b) or (3.6) remains unchanged, but the upper bound takes the following form, with E^P denoting the expectation under the actual return distribution $P(z_{t+\Delta t} | S_t)$:

$$\overline{C}_T(S_T) = (S_T - K)^+, \; \overline{C}_t(S_t) = \frac{E^P[\overline{C}_{t+\Delta t}(S_t(1+z_{t+\Delta t})) | S_t]}{E[1+z_{t+\Delta t} | S_t]}. \tag{3.11}$$

When the returns are iid then (3.11) corresponds to the expected payoff given S_t discounted by the risky asset's return. The upper bound of (3.11) has been extended to allow for proportional transaction costs. The same is true for the lower bound given by (3.6).[26]

It can be easily seen from both (3.5ab) and (3.6) that the distributions are risk neutral, with $E^{U_t}(1+z_{t+\Delta t}) = E^{L_t}(1+z_{t+\Delta t}) = R$. These distributions were derived from the LP in (3.2), and are independent of option characteristics such as the strike price or time to expiration. Note also that the pricing kernel $Y(z_{t+\Delta t})$ corresponding to the upper bound has a "spike" at $z_{\min,t+\Delta t}$ and is constant thereafter, while the kernel of the lower bound is constant and positive till a value z_t^* such that $E[(1+z_{t+\Delta t}) | z_{t+\Delta t} \le z_t^*] = R$, and becomes zero for $z_{t+\Delta t} > z_t^*$. These pricing kernels are boundary marginal utilities that do not correspond to a CPRA utility function or, indeed, to any class of utility functions with continuously decreasing marginal utilities.

The distributions U_t and L_t are the incomplete market counterparts of the risk neutral probabilities of the binomial model, the only discrete time complete market model. If, in addition to payoff convexity, the underlying asset returns are iid then U_t and L_t are time-independent and independent of the stock price S_t. In all cases, however, the distributions U_t and L_t depend on the entire actual distribution of the underlying asset, and not only on its volatility parameter, as in the binomial and the BSM models. In particular, they depend on the mean $\hat{z}$ of the distribution. If $1+\hat{z} = R$ then (3.5ab) and (3.6) imply that the two distributions U_t and L_t coincide. As $\hat{z}$ increases above $R-1$ the bounds widen, reflecting the incompleteness of the market. The dependence of U_t and L_t on convexity and on the entire return distribution may appear restrictive, but in fact the approach is quite general. The stochastic dominance assumptions may still be used to find the tightest bounds that can be supported by the market equilibrium monotonicity condition by solving the LP (3.2) recursively when convexity does not hold, with the bounds now depending in general on option characteristics. Recall that arbitrage and equilibrium models are able to provide expressions for option prices only under specific assumptions about asset dynamics. By contrast the stochastic dominance approach can accommodate any type of asset dynamics, including time- and state-varying distributions, provided a suitable discrete time representation similar to (2.2) can be found. As shown in the next section, the dependence on many parameters of the distribution, including $\hat{z}$ in the

[26] See Propositions 1 and 5 of Constantinides and Perrakis (2002).

diffusion case, disappears at the continuous time limit.

We provide a numerical example of the width of the bounds for a case in which the underlying asset return is given by (2.1), with $\lambda_t = 0$ and with mean and volatility parameters similar to those normally prevailing in the market for the S&P 500 index. Let $S_t = 100$, $T = 0.5$, $r = 4\%, \mu = 8\%, \sigma = 20\%$ in (2.1), with the parameter $\varepsilon \sim$ uniform in $[-\sqrt{3}, \sqrt{3}]$, and consider an at-the-money call option with one period to expiration. The option is in the money for $\varepsilon \geq -\frac{0.2}{\sqrt{0.5}}$. Applying now equations (3.6) we see that the upper bound is equal to the expected payoff discounted by the riskless rate 1.02 and multiplied by the probability $1-Q$, yielding $\overline{C} = 7.772$. For the lower bound we first identify the value $\varepsilon^* = 1.4492$ such that $E(1 + \mu T + \sigma\varepsilon\sqrt{T} \,|\varepsilon \leq \varepsilon^*) = R = 1.02$ and then we find the lower bound as the conditional payoff expectation given $\varepsilon \leq \varepsilon^*$ discounted by 1.02, yielding $\underline{C} = 6.537$. These correspond to a width of 17.26% of the midpoint for these single period bounds, which is expected to decrease in the presence of intermediate trading.[27] In the next section we explore the limits of the expressions in (3.5ab) and (3.6) when $z_{t+\Delta t}$ is given by the continuous time processes (2.8) and (2.9).

IV. <u>Option Pricing for Diffusion and Jump-Diffusion Processes</u>

The recursive procedure described in (3.4) and (3.6) can be applied directly to the stock returns $z_{t+\Delta t}$ given by (2.1) or by (2.2) in order to generate the upper and lower bounds at time zero. Of particular interest, however, is the existence of a limit to these bounds as $\Delta t \to 0$ and (2.1) tends to (2.9). These limits are expressed by the following proposition whose proof is in the appendix. Although this proposition does not contain any new option pricing results, it does provide a link between the option bounds approach and the continuous time results that underlie most of option pricing. It is also necessary for the proof of the jump-diffusion results in Propositions 3 and 4, which are novel.

<u>Proposition 2</u>: *When the underlying asset follows a continuous time process described by the diffusion (2.8) then both upper and lower bounds (3.4)-(3.6) of a European call option evaluated on the basis of the discretization of the returns given by (2.1) converge to the same value, equal to the expectation of the terminal payoff of an option on an asset whose dynamics are described by the process*

$$\frac{dS_t}{S_t} = rdt + \sigma(S_t, t)dW \,, \tag{4.1}$$

[27] Numerical results for the bounds are also in Ritchken (1985) for single period lognormal returns and in Ritchken and Kuo (1988) for multiperiod trinomial returns. An empirical application of the bound (3.11) under transaction costs to S&P 500 index options is in Constantinides, Jackwerth and Perrakis (2009).

discounted by the riskless rate.

This result establishes the formal equivalence of the bounds approach to the prevailing arbitrage methodology for plain vanilla option prices whenever the underlying asset dynamics are generated by a diffusion or Ito process, no matter how complex. Note that the univariate Ito process is the only type of asset dynamics, corresponding to dynamically complete markets, for which options can be priced by arbitrage considerations alone. The two bounds (3.4)-(3.6), therefore, by defining the admissible set of option prices for any discrete time distribution corresponding to such a dynamic completeness, generalize the binomial model to any type of discrete time distribution.

From the proof of Proposition 2 it is clear that the result holds because at the limit the pricing kernels of both upper and lower bound continue to play their risk-neutralizing role, while their effect on the instantaneous variance of the process disappears.[28] This property *does not* extend to the jump-diffusion case, as shown further on in this section.

(Figure 4.1 about here)

Figure 4.1 illustrates the convergence of the two bounds to the BSM value for an at-the-money call option with K = 100 and T = 0.25 years for the following instantaneous annual parameters: $r = 3\%, \mu = 5\%$ to $9\%, \sigma = 10\%$. The diffusion process was approximated by a 300-period trinomial tree constructed according to the algorithm of Kamrad and Ritchken (1991). The two option bounds were evaluated as discounted expectations of the payoffs under the risk neutral probabilities obtained by applying the expressions (3.4)-(3.6) to subtrees of the 300-period trinomial tree. Fast Fourrier Transforms were applied for the derivation of the terminal distributions of the underlying asset and the bounds, given that the returns are iid.

As the figure shows, the two bounds converge to their common limit uniformly from below and above respectively. The speed of convergence varies inversely with the size of the risk premium, but convergence is essentially complete after 300 periods even for the largest premium of 6%. This speed of convergence may also be helpful for the cases where no closed-form expression for the option price exists, as in complex cases of state-dependent univariate diffusions, like the Constant Elasticity of Variance (CEV) model.[29] In such cases valuation of the option by Monte Carlo simulation of the bounds is

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[28] Mathur and Ritchken (1999) show that the BSM option price arises as the *lower bound* of the LP program (3.2) in a single period model, in which the pricing kernel has been restricted to satisfy the Decreasing Relative Risk Aversion (DRRA) property; this bound does not tighten when the time interval is subdivided. In our case the discrete time lower bound of the LP is always lower than the one corresponding to DRRA since the class of admissible kernels has been restricted, but it tightens with denser subdivisions and becomes equal to the DRRA and the BSM model at the limit.

[29] See Cox and Rubinstein (1985, pp. 361-364).

certainly an alternative to an option value computed as a discounted payoff of paths generated by the Monte Carlo simulation of (4.1). While there may not be any computational advantages in going through the bounds route to option valuation, the fact that both upper and lower bound tend to the same limit from above and from below may provide a benchmark for the accuracy of the valuation, in contrast to the direct simulation of (4.1).

Next we examine the limiting behavior of the stochastic dominance bounds that can be derived from the discrete time process (2.2) that was shown by Lemma 2 to tend to a jump-diffusion. For such a process a unique option price can be derived by arbitrage methods alone only if $\sigma = 0$ and J takes exactly one value when a jump occurs. In such a case the process (2.2) is binomial and it can be readily verified that the distributions U_t and L_t coincide, and the stochastic dominance approach yields the same unique option price as the binomial jump process in Cox, Ross and Rubinstein (1979). Otherwise, we must examine the two bounds separately. For the option upper bound we apply the transformation (3.6) to the discretization (2.2), taking into account that the variable J takes both positive and negative values, or that $J_{\min} < 0 < J_{\max}$. For such a process we note that as Δt decreases, there exists h, such that for any $\Delta t \le h$, the minimum outcome of the jump component is less than the minimum outcome of the diffusion component, $J_{\min} < \mu_t \Delta t + \sigma_t \varepsilon_{\min} \sqrt{\Delta t}$. Consequently, for any $\Delta t \le h$, the minimum outcome of the returns distribution is $J_{\min}$, which is the value that we substitute for $z_{\min,t+\Delta t}$ in (3.6). With such a substitution we have now the following result, proven in the appendix.

<u>Proposition 3</u>: *When the underlying asset follows a jump-diffusion process described by (2.9) the upper option bound is the expected payoff discounted by the riskless rate of an option on an asset whose dynamics are described by the jump-diffusion process*

$$\frac{dS_t}{S_t} = \left[r - (\lambda + \lambda_U) \mu_{Jt}^U \right] dt + \sigma_t dW + J_t^U dN \tag{4.2}$$

where,

$$\lambda_{Ut} = -\frac{\mu_t - r}{J_{\min}} \tag{4.3}$$

and J_t^U *is a mixture of jumps with intensity* $\lambda + \lambda_{Ut}$ *and distribution and mean*

$$\begin{aligned} J_t^U &= \begin{cases} J & \text{with probability } \frac{\lambda}{\lambda+\lambda_{Ut}} \\ J_{\min} & \text{with probability } \frac{\lambda_{Ut}}{\lambda+\lambda_{Ut}} \end{cases} \\ \mu_{Jt}^U &= \frac{\lambda}{\lambda+\lambda_{Ut}} \mu_J + \frac{\lambda_{Ut}}{\lambda+\lambda_{Ut}} J_{\min} \end{aligned} \tag{4.4}$$

By definition of the convergence of the discrete time process, Proposition 3 states that the call upper bound is the discounted expectation of the call payoff under the jump-diffusion process given by (4.2), which implies that the transformed jump component J_t^U is fully diversifiable. We may, therefore, use the results derived by Merton (1976) for options on assets following jump-diffusion processes with the jump risk fully diversifiable.[30] Applying Merton's approach to (4.2) we find that the upper bound on call option prices for the jump-diffusion process (2.9) must satisfy the following partial differential equation (pde), with terminal condition $C(S_T, T) = \max\{S_T - K, 0\}$:

$$\left[r-(\lambda+\lambda_{Ut})\mu_{Jt}^U\right] S\frac{\partial \overline{C}}{\partial S} - \frac{\partial \overline{C}}{\partial T} + \frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 \overline{C}}{\partial S^2} + (\lambda+\lambda_{Ut})E^U[\overline{C}(S(1+J_t^U)) - \overline{C}(S)] - r\overline{C} = 0 . \tag{4.5}$$

An important special case is when the lower limit of the jump amplitude is equal to 0, in which case $J_{\min} = -1$ and the return distribution has an absorbing state in which the stock becomes worthless and $z_{1t+\Delta t} = z_{\min,t+\Delta t} = -1$; this is the case described in (3.11), in which as we saw the option price is the expected payoff with the actual distribution, discounted by the expected return on the stock. Hence, this is identical to the Merton (1976, equation (14)) case with *r* replaced by μ_t , yielding

$$\left[\mu_t - \lambda\mu_J\right] S\frac{\partial \overline{C}}{\partial S} - \frac{\partial \overline{C}}{\partial T} + \frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 \overline{C}}{\partial S^2} + \lambda E[\overline{C}(S(1+J)) - \overline{C}(S)] - \mu_t \overline{C} = 0 \tag{4.6}$$

If (4.6) holds and we assume, in addition, that the diffusion parameters are constant and the amplitude of the jumps has a lognormal distribution with $\ln(1+J) \sim N(\mu_J, \sigma_J)$, the distribution of the asset price given that *k* jumps occurred is conditionally normal, with mean and variance

$$\begin{aligned} \mu_k &= \mu - k\lambda\mu_J + \frac{k}{T}\ln(1+\mu_J) \\ \sigma_k^2 &= \sigma^2 + \frac{k}{T}\sigma_J^2 \end{aligned} \tag{4.7}$$

Hence, if *k* jumps occurred, the option price would be a Black-Scholes expression with μ_k replacing the riskless rate *r*, or $BS(S, K, T, \mu_k, \sigma_k)$. Integrating (4.6) would then yield the following upper bound, which can be obtained directly from Merton (1976) by replacing *r* by μ .

$$\overline{C} = \sum_{k=0}^{\infty} \exp[-\lambda(1+\mu_J)T] \frac{[\lambda(1+\mu_J)T]^k}{k!} BS(S, K, T, \mu_k, \sigma_k) \tag{4.8}$$

When the jump distribution is not normal, the conditional asset distribution given k jumps

[30] Note that we do not assume here that the jump risk is diversifiable.

is the convolution of a normal and k jump distributions. The upper bound cannot be obtained in closed form, but it is possible to obtain the characteristic function of the bound distribution. Similar approaches can be applied to the integration of equation (4.5), which holds whenever $0 > J_{\min} > -1$. Closed form solutions can also be found whenever the amplitude of the jumps is fixed as, for instance, when there is only an up and a down jump of a fixed size.[31] A pde similar to (4.5) also holds if the process has only "up" jumps, in which case $J_{\min} = 0$ and the lowest return $z_{\min}$ in (3.6) comes from the diffusion component. In such a case the key probability Q of (3.6) is the same as in the proof of Proposition 2 and (4.5) still holds with $\lambda_{Ut} = 0$, implying that the option upper bound is the Merton (1976) bound, with the jump risk fully diversifiable.

The option lower bound for the jump-diffusion process given by (2.9) and its discretization (2.2) is found by a similar procedure. We apply $L(z_{t+\Delta t})$ from (3.6) to the process (2.2) and we prove in the appendix the following result.

Proposition 4*: *When the underlying asset follows a jump-diffusion process described by (2.9) , the lower option bound is the expected payoff discounted by the riskless rate of an option on an asset whose dynamics is described by the jump-diffusion process

$$\frac{dS_t}{S_t} = \left[r - \lambda \mu_{Jt}^L \right] dt + \sigma_t dW + J_t^L dN \tag{4.9}$$

where J_t^L is a jump with the truncated distribution

$$J \mid J \le \overline{J}_t$$

The mean μ_{Jt}^L of the jump and the value of $\overline{J}_t$ can be obtained by solving the equations

$$\begin{aligned} &\mu_t - \lambda \mu_J + \lambda \mu_{Jt}^L = r \\ &\mu_{Jt}^L = E(J \mid J \le \overline{J}_t) \end{aligned} \tag{4.10}$$

Observe that (4.10) always has a solution since $\mu_t > r$ by assumption. The limiting distribution includes the whole diffusion component and a truncated jump component. Unlike simple diffusion, the truncation does not disappear as $\Delta t \to 0$. As with the upper bound, we can apply the Merton (1976) approach to derive the pde satisfied by the option lower bound, which is given by

$$\left[r - \lambda \mu_{Jt}^L \right] S \frac{\partial \underline{C}}{\partial S} - \frac{\partial \underline{C}}{\partial T} + \frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 \underline{C}}{\partial S^2} + \lambda E^L[\underline{C}(S(1+J_t^L)) - \underline{C}(S)] - r\underline{C} = 0 \tag{4.11}$$

with terminal condition $\underline{C}_T = C(S_T, T) = \max\{S_T - K, 0\}$. The solution of (4.11) can be obtained in closed form only when the jump amplitudes are fixed, since even when the

[31] See, for instance, the example in Masson and Perrakis (2000).

jumps are normally distributed, the lower bound jump distribution is truncated.

Observe that the jump components in both $\overline{C_t}(S_t)$ and $\underline{C}_t(S_t)$ are now state-dependent if μ_t, the diffusion component of the instantaneous expected return on the stock, is state-dependent., even though the actual jump process is independent of the diffusion. In many empirical applications of jump-diffusion processes, which were on the S&P 500 index options, the unconditional estimates are considered unreliable. On the other hand there is consensus that the unconditional mean is in the 4-6% range;[32] this is reflected in the numerical results below. Observe also that for normally distributed jumps the only parameters that enter into the computation of the bounds are the mean of the process, the volatility of the diffusion and the parameters of the jump component. Hence, the information requirements are the same as in the more traditional approaches, with the important difference that the mean of the process replaces the risk aversion parameter.

We present in Table 4.1 and Figure 4.2 estimates of the bounds under a jump-diffusion process for an at-the-money option with K = 100 and maturity T = 0.25 years for varying subdivisions of the time to expiration, and with the following annual parameters: $r = 3\%, \mu = 5\%$ to $9\%, \sigma = 10\%, \lambda = 0.3, \mu_J = -0.05, \sigma_J = 7\%$. Table 4.2 and Figure 4.3 present the bounds for $\mu = 7\%$ and with the other parameters unchanged for various degrees of moneyness of the option. The jump-diffusion process was approximated by a 300-time step tree built according to the method introduced by Amin (1993). The jump amplitude distribution was lognormal, which was truncated for numerical purposes in building the tree. The bounds were computed by taking the discounted expectations of the payoff under the time-varying risk neutral probabilities of (3.4) applied to subtrees. The risk neutral price is the Merton (1976) price for this process.

(Table 4.1 about here)

(Figure 4.2 about here)

(Table 4.2 about here)

(Figure 4.3 about here)

The results shown in Table 4.1 show a maximum spread between bounds of about 10%, a spread that is an increasing function of μ. In Table 4.2 the spread is much lower for in-the-money options and reaches about 18% for the most out-of-the money options. Note

[32] See Fama and French (2002), Constantinides (2002) and Dimson, Marsh and Staunton (2006).

that the range of values of μ implies an *ex-dividend* risk premium range from 2% to 7%, a range that covers what most people would consider the appropriate value of such a premium in many important cases; the corresponding width of the bounds ranges from 4.5% to 10%. For the most commonly chosen risk premium of 4%, corresponding to $\mu = 7\%$, the spread at-the-money is about 7.26%. This range of allowable option prices in the stochastic dominance approach is the exact counterpart of the inability of the "traditional" arbitrage-based approaches to produce a single option price for jump diffusion processes without an arbitrarily chosen risk aversion parameter, even when the models have been augmented in this case by general equilibrium considerations.

V. Extensions to Stock Options

In this section we extend the set of assumptions about market equilibrium, by defining the risky asset held by the trader as a market index portfolio and allowing the trader to adopt additional marginal positions in a single stock, as well as in options on that stock. Let now I_t denote the current value of the index and S_t the value of the stock, with the returns $\frac{I_{t+\Delta t} - I_t}{I_t} \equiv z_{t+\Delta t}$ and $\frac{S_{t+\Delta t} - S_t}{S_t} \equiv v_{t+\Delta t}$. The market equilibrium conditions (3.1) are now as follows

$$\begin{aligned} &E[Y(z_{t+\Delta t}) | I_t, S_t] = R^{-1}, \\ &E[(1+z_{t+\Delta t})Y(z_{t+\Delta t}) | I_t, S_t] = E[(1+v_{t+\Delta t})Y(z_{t+\Delta t}) | I_t, S_t] = 1, \\ &C_t(S_t, I_t) = E[C_{t+\Delta t}(S_t(1+v_{t+\Delta t}), I_t(1+z_{t+\Delta t}))Y(z_{t+\Delta t}) | I_t, S_t] \end{aligned} \tag{5.1}$$

Assume now a joint discrete distribution of the two returns, and set $E[v_{t+\Delta t} | z_{t+\Delta t} = z_{jt+\Delta t}, S_t, I_t] = \bar{v}_j(z_{jt+\Delta t}) \equiv \bar{v}_{jt+\Delta t}$. The equilibrium relations (5.1) imply certain restrictions on the parameters of the joint distribution. These are expressed by the following Lemma. It covers the case of diffusion and can be extended to cover jump-diffusion for that joint return distribution.

Lemma 4: *If the function $\bar{\nu}_j(z_{jt+\Delta t})$ is linear, $\bar{\nu}_j(z_{jt+\Delta t}) = \theta + \zeta z_{jt+\Delta t}$, then the following relation must hold:*

$$(R-1)(1-\zeta) = \theta. \tag{5.2}$$

Further, if in addition $E[C_{t+\Delta t}(S_t(1+v_{t+\Delta t}), I_t(1+z_{t+\Delta t})) | z_{t+\Delta t}]$ can be written as a function $\hat{C}_{t+\Delta t}(S_t(1+\bar{v}(z_{t+\Delta t})), I_t(1+z_{t+\Delta t}))$ then $C_t(S_t, I_t)$ takes the form $C_t(S_t)$, independent of the index level I_t.

Proof: We write the last relation in (5.1)

$E[E[(1+v_{t+\Delta t})Y(z_{t+\Delta t})\big|z_{t+\Delta t}]\big|I_t,S_t]=E[(1+\bar{v}(z_{t+\Delta t}))Y(z_{t+\Delta t})\big|I_t,S_t]=1$. (5.2) then follows directly from the first two relations in (5.1) by replacing $\bar{\nu}(z_{t+\Delta t})$. For the second part of the Lemma, we use induction. Since the Lemma obviously holds at T-1, we assume that it holds at $t+\Delta t$ and apply the last relation of (5.1) to $\hat{C}_{t+\Delta t}(S_t(1+\bar{v}(z_{t+\Delta t})))$, QED.

Assume now that Lemma 4 holds and define

$$
\begin{aligned}
\bar{\hat{C}}_T(S_T) &= \underline{\hat{C}}_T(S_T)=(S_T-K)^+ \\
\bar{C}_{t+\Delta t}(S_t,z_{t+\Delta t}) &\equiv E^P[\bar{\hat{C}}_{t+\Delta t}(S_t(1+v_{t+\Delta t}))\big|z_{t+\Delta t}],\ t\le T-1 \\
\underline{C}_{t+\Delta t}(S_t,z_{t+\Delta t}) &\equiv E^P[\underline{\hat{C}}_{t+\Delta t}(S_t(1+v_{t+\Delta t}))\big|z_{t+\Delta t}],\ t\le T-1. \\
\bar{\hat{C}}_t(S_t) &= \frac{1}{R}E^{U_t}\{\bar{C}_{t+\Delta t}(S_t,z_{t+\Delta t})\big|S_t\} \\
\underline{\hat{C}}_t(S_t) &= \frac{1}{R}E^{L_t}\{\underline{C}_{t+\Delta t}(S_t,z_{t+\Delta t})\big|S_t\}
\end{aligned}
\tag{5.3}
$$

The distributions U_t, L_t are those given by (3.5) or (3.6) with P denoting the original distribution. The following result extends Lemma 3 to stock options.

Lemma 5: *Under the conditions of Lemma 4, if the option price* $C_t(S_t)$ *is convex in* S_t *then the relations (5.3) define bounds such that* $\underline{\hat{C}}_t(S_t)\le C_t(S_t)\le \bar{\hat{C}}_t(S_t)$.

Proof: We show that Lemma 3 holds, using again the convexity of $C_t(S_t)$. At any $t\le T-1$ we multiply $C_{t+\Delta t}(S_t(1+v_{t+\Delta t}))$ by $Y(z_{t+\Delta t})$ and take conditional expectations given $z_{t+\Delta t}$. The bounds on $C_t(S_t)$ are then found by solving the following LP, which replaces (3.2)

$$
\begin{aligned}
&\max_{Y_{jt+\Delta t}} \sum_{j=1}^{j=n}\hat{C}_{t+\Delta t}(S_t(1+\bar{v}(z_{jt+\Delta t})))p_{jt+\Delta t}Y_{jt+\Delta t} \\
&(\min_{Y_{jt+\Delta t}} \sum_{j=1}^{j=n}\hat{C}_{t+\Delta t}(S_t(1+\bar{v}(z_{jt+\Delta t})))p_{jt+\Delta t}Y_{jt+\Delta t})
\end{aligned}
, \tag{5.4}
$$

subject to the same constraints as (3.2) *plus* the additional constraint $1=\sum_{j=1}^{n}(1+\bar{v}_j(z_{jt+\Delta t}))p_{jt+\Delta t}Y_{jt+\Delta t}$. If, however, the linearity condition of Lemma 4 holds then this last constraint is redundant and any feasible solution of the LP (3.2) with the modified objective function satisfies also this additional constraint. Since the linearity condition also implies that $\hat{C}_{t+\Delta t}(S_t(1+\bar{v}(z_{t+\Delta t})))$ is convex in $z_{t+\Delta t}$, the bounds of $C_t(S_t)$ are found by taking expectations of $C_{t+\Delta t}(S_t(1+v_{t+\Delta t}))$ with respect to the distributions U_t, L_t

given by (3.5) or (3.6), QED.[33] The proof of Lemma 5 then follows directly by using induction, as in the proof of Proposition 1, QED.

We examine separately diffusion and jump-diffusion for both the index and the stock. For diffusion Proposition 5 shows that both bounds converge to the BSM option price in this case as well, as with index options. Define

$$\begin{aligned}
& z_{t+\Delta t} = \mu_t \Delta t + \sigma_t \varepsilon \sqrt{\Delta t} \\
& v_{t+\Delta t} = m_t \Delta t + \sigma_t^v (\rho\varepsilon + \eta\sqrt{1-\rho^2})\sqrt{\Delta t}, \\
& E(\varepsilon) = E(\eta) = E(\varepsilon\eta) = 0
\end{aligned} \tag{5.5}$$

with $\varepsilon \sim D_z(0,1)$ and $\eta \sim D_v(0,1)$. It is clear that Lemma 1 holds and both index and stock converge to the following bivariate diffusion

$$\frac{dI_t}{I_t} = \mu_t dt + \sigma_t dW_1, \qquad \frac{dS_t}{S_t} = m_t dt + \sigma_t^v dW_2, \tag{5.6}$$

where μ_t, m_t are the instantaneous means, σ_t, σ_t^v the corresponding volatilities, and $E[dW_1 dW_2] = \rho dt$. Further, Lemma 4 holds and

$$\overline{\nu}_j(z_{jt+\Delta t}) = (m_t - \mu_t \frac{\rho\sigma_t^\nu}{\sigma_t})\Delta t + \frac{\rho\sigma_t^\nu}{\sigma_t} z_{jt+\Delta t} = m_t \Delta t + \rho\sigma_t^\nu \varepsilon_j \sqrt{\Delta t}. \tag{5.7}$$

We then have the following result, proven in the appendix.

<u>Proposition 5</u>: *When both the index and the underlying asset follow continuous time processes described by the bivariate diffusion (5.6) then both upper and lower bounds (3.4)-(3.6) of a European call option evaluated on the basis of the discretization of the returns given by (5.5) converge to the same value, equal to the expectation of the terminal payoff of an option on an asset whose dynamics are described by the process*

$$\frac{dS_t}{S_t} = r dt + \sigma_t^v dW. \tag{5.8}$$

Next we examine the jump-diffusion case. We model it as a mixture, in which the diffusion returns $z_{D,t+\Delta t}$ and $v_{D,t+\Delta t}$ occur with probability $1-\lambda_t \Delta t$ as given by

[33] In evaluating U_t, L_t the terms $\overline{\nu}(\hat{z}_{t+\Delta t})$ and $\overline{\nu}(z_{\min,t+\Delta t})$ should replace $\hat{z}_{t+\Delta t}$ and $z_{\min,t+\Delta t}$.

$$
\begin{aligned}
z_{D,t+\Delta t} &= (\mu_t - \lambda\mu_{JI})\Delta t + \sigma_t \varepsilon \sqrt{\Delta t} \\
v_{D,t+\Delta t} &= (m_t - \lambda\mu_{JS})\Delta t + \sigma_t^v (\rho\varepsilon + \eta\sqrt{1-\rho^2})\sqrt{\Delta t}
\end{aligned} . \tag{5.11}
$$

With probability $\lambda_t \Delta t$ there are correlated jumps J_{It} and J_{St} in both index and stock returns $z_{J,t+\Delta t}$ and $v_{J,t+\Delta t}$. We shall again adopt the assumption that the jump process is state- and time-independent, with $\lambda_t = \lambda,\ J_{It} = J_I,\ J_{St} = J_S$. The jumps are random variables with means μ_{JI} and μ_{JS}, and standard deviations σ_{JI} and σ_{JS}. We model the mutual dependence of the jumps by setting

$$
\ln(1+J_I) = \mu_{JI} + \sigma_{JI}\mathrm{X}, \quad \ln(1+J_S) = \mu_{JS} + \psi\sigma_{JS}\mathrm{X} + \sqrt{1-\psi^2}\sigma_{JS}\Psi , \tag{5.12}
$$

where the independent random variables X and Ψ have distributions $D_{J\mathrm{X}}$ and $D_{J\Psi}$ with mean 0 and variance 1, not necessarily normal. As with the index options, we assume that $-1 \le J_{I\min} = \exp(\mu_{JI} + \sigma_{JI}\mathrm{X}_{\min}) - 1 < 0 < J_{I\max}$, $-1 \le J_{S\min} < 0 < J_{S\max}$, and we similarly define

$$
\ln(1+J_{SI}) \equiv E[\mu_{JS} + \psi\sigma_{JS}\mathrm{X} + \sqrt{1-\psi^2}\sigma_{JS}\Psi) \,|\, \mathrm{X}] = \mu_{JS} + \psi\sigma_{JS}\mathrm{X} , \tag{5.13}
$$

with $-1 \le J_{SI\min} = \exp(\mu_{JS} + \psi\sigma_{JS}\mathrm{X}_{\min}) - 1$. Lemma 2 obviously applies to these discretizations as well, and both index and stock returns tend to jump-diffusion asset dynamics similar to (2.9).

$$
\frac{dI_t}{I_t} = (\mu_t - \lambda\mu_{JI})dt + \sigma_t dW_1 + J_I dN, \quad \frac{dS_t}{S_t} = (m_t - \lambda\mu_{JS})dt + \sigma_t^v dW_2 + J_S dN, \tag{5.14}
$$

with $E(dW_1 dW_2) = \rho dt$ and the joint distribution of the amplitudes given by (5.12). Similarly, Lemma 4 holds here as well, since both $v_{D,t+\Delta t}$ and $v_{J,t+\Delta t}$ are linear functions of $z_{D,t+\Delta t}$ and $z_{J,t+\Delta t}$ respectively. The following result, proven in the appendix, extends Propositions 3 and 4 to stock options.

Proposition 6: *When both the index and the underlying asset follow a jump-diffusion process described by (5.13) the upper option bound is the expected payoff discounted by the riskless rate of an option on an asset whose dynamics are described by the jump-diffusion process*

$$
\begin{aligned}
&\frac{dS_t}{S_t} = [r - (\lambda + \lambda_{Ut})\mu_{tJS}^U]dt + \sigma_t^v dW_2 + J_{tS}^U dN, \ \lambda_{Ut} = -\frac{(m_t - r)}{J_{SI\min}}, \\
&\mu_{tJS}^U = \frac{\lambda}{\lambda + \lambda_{Ut}}\mu_{JS} + \frac{\lambda_{Ut}}{\lambda + \lambda_{Ut}} J_{SI\min}, \ J_{tS}^U = \begin{cases} J_S & \text{with probability } \frac{\lambda}{\lambda+\lambda_{Ut}} \\ J_{SI\min} & \text{with probability } \frac{\lambda_{Ut}}{\lambda+\lambda_{Ut}} \end{cases}
\end{aligned} \tag{5.15}
$$

As for the lower option bound, it is the expected payoff discounted by the riskless rate of *an option on an asset whose dynamics are described by the jump-diffusion process*

$$\frac{dS_t}{S_t} = \left[r - \lambda \mu_{tJS}^L \right] dt + \sigma_t^v dW_2 + J_{tS}^L dN, \tag{5.16}$$

where J_{tS}^L *is a jump process defined from (5.12)-(5.13) as follows*

$$\begin{aligned} &\ln(1 + J_{tS}^L) = \mu_{tJS}^L + \psi \sigma_{JS} \hat{\mathrm{X}}^t + \sqrt{1 - \psi^2} \sigma_{JS} \Psi, \\ &\hat{\mathrm{X}}^{\mathrm{t}} = \mathrm{X} \middle| \mathrm{X} \le \overline{\mathrm{X}}^t, \;\; m_t - \lambda \mu_{JS} + \lambda \mu_{tJS}^L = r \end{aligned} \quad . \tag{5.17}$$

Proposition 6 yields the bounds for the stock options as functions of the parameters of the joint jump distribution of the index and the stock, as well as the volatility and mean of the stock diffusion component. With the exception of the mean, these are the same parameters as the ones required to value options in the conventional equilibrium-based models. Unlike these models, the bounds in Proposition 6 do not require a lognormal distribution of the jump amplitude. As with the index options, both upper and lower bound can be expressed as the solutions of pde's similar to (4.5) and (4.11), while the upper bound is available in closed form with an equation similar to (4.8) whenever the distribution of the jump amplitude is lognormal.

VI. Extensions and Conclusions

The results presented in the previous section yield bounds for jump-diffusion option prices that are relatively simple to compute and reasonably tight for most empirically important cases. The alternative approach of using an assumed value of the risk aversion parameter to price the option induces, as can be easily verified, a much wider range of admissible option prices if that parameter is allowed to vary over its relevant range from 1 to more than 40. The bounds can also accommodate state-dependent diffusion parameters, even though their computation would be difficult. If the upper (lower) option bound is violated by observed market prices then there exist portfolios involving the option, the stock and the riskless asset that improve the expected utility of any risk averse trader by adopting a short (long) position in the mispriced option.[34]

Computational difficulties are also likely to arise in the main extension of the jump-diffusion option models presented in this paper, the incorporation of stochastic volatility (SV) into the stock returns. This introduces an additional source of systematic risk, which can be handled either by arbitrage or by equilibrium considerations. We sketch below an extension of our approach to the pricing of jump risk that can incorporate SV, provided its systematic risk implications are handled outside our model.[35]

[34] These portfolios are evaluated for both diffusion and jump-diffusion in an appendix, available from the authors on request.
[35] See the discussion in Oancea and Perrakis (2007).

In a combined SV and jump-diffusion process the stock returns are still given by (2.9) but the volatility σ_t is random and follows a general diffusion, often a mean-reverting Ornstein-Uhlenbeck process.[36] In our case we use a general form with an unspecified instantaneous mean $m(\sigma_t^2)$ and volatility $s(\sigma_t^2)$. The asset dynamics then become

$$\begin{aligned} \frac{dS_t}{S_t} &= (\mu_t - \lambda\mu_J)dt + \sigma_t dW_1 + JdN \\ d\sigma_t^2 &= m(\sigma_t^2)dt + s(\sigma_t^2)dW_2, \quad dW_1 dW_2 = \rho(\sigma_t^2)dt \end{aligned} \tag{6.1}$$

The following discrete representation can be easily shown by applying Lemmas 1 and 2 to converge to (5.1):[37]

$$\begin{aligned} (S_{t+\Delta t} - S_t)/S_t &\equiv z_{t+\Delta t} = \mu(S_t)\Delta t + \sigma_t \varepsilon\sqrt{\Delta t} + J\Delta N \\ \sigma_{t+\Delta t}^2 - \sigma_t^2 &= m(\sigma_t^2)\Delta t + s(\sigma_t^2)\eta\sqrt{\Delta t} \end{aligned}, \tag{6.2}$$

where η is an error term of mean 0 and variance 1, and with correlation $\rho(\sigma_t^2)$ between ε and η. In what follows we shall assume that this correlation is constant.

Under reasonable regularity conditions the pricing kernel at time t conditional on the state variable vector (S_t, σ_t) is monotone decreasing. Similarly, for any given σ_t the option price is convex in the stock price.[38] Hence, for any given volatility path over the interval $[0, T]$ to option expiration the option prices at any time t are bound by the expressions $\overline{C}_t(S_t, \sigma_t)$ and $\underline{C}_t(S_t, \sigma_t)$ given in (3.5ab)-(3.7). Since both of these expressions are expected option payoffs under risk neutral distributions, we can apply arbitrage methods as in Merton (1976) to price the options *given a price* $\xi(S_t, \sigma_t, t)$ for the volatility risk. Proposition 3 and 4, therefore, hold and the admissible option bounds satisfy the following pde's:

$$\begin{aligned} &\left[r - (\lambda + \lambda_{Ut})\mu_{Jt}^U\right] S\frac{\partial \overline{C}}{\partial S} - \frac{\partial \overline{C}}{\partial T} + \frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 \overline{C}}{\partial S^2} + \rho\sigma_t s(\sigma_t^2)\frac{\partial^2 \overline{C}}{\partial S \partial \sigma_t^2} + \frac{1}{2}s^2(\sigma_t^2)\frac{\partial^2 \overline{C}}{\partial^2 \sigma_t^2} \\ &+ \frac{\partial \overline{C}}{\partial \sigma_t^2}[m(\sigma_t^2) - \xi(S_t, \sigma_t, t)] + (\lambda + \lambda_{Ut})E^U[\overline{C}(S(1 + J_t^U)) - \overline{C}(S)] - r\overline{C} = 0 \end{aligned} \tag{6.3}$$

[36] See Heston (1993).

[37] In (2.3)-(2.7) the vector ϕ_t in applying the Lindeberg condition is now two-dimensional, (S_t, σ_t^2).

[38] See the results of Bergman, Grundy and Wiener (1996) for a bivariate diffusion.

$$
\begin{aligned}
&\left[r-\lambda\mu_{Jt}^{L}\right]S\frac{\partial\underline{C}}{\partial S}-\frac{\partial\underline{C}}{\partial T}+\frac{1}{2}\sigma_t^2S^2\frac{\partial^2\underline{C}}{\partial S^2}+\rho\sigma_t s(\sigma_t^2)\frac{\partial^2\underline{C}}{\partial S\partial\sigma_t^2}\\
&+\frac{1}{2}s^2(\sigma_t^2)\frac{\partial^2\underline{C}}{\partial^2\sigma_t^2}+\frac{\partial\underline{C}}{\partial\sigma_t^2}[m(\sigma_t^2)-\xi(S_t,\sigma_t,t)]+\lambda E^L[\underline{C}(S(1+J_t^L))-\underline{C}(S)]-r\underline{C}=0
\end{aligned}
\tag{6.4}
$$

The estimation of (6.3)-(6.4) under general conditions presents computational challenges that lie outside the scope of this paper.

Last, the discrete time approach of the bounds estimation allows two significant extensions to jump-diffusion option pricing: the valuation of American options and the incorporation of proportional transaction costs in trading the underlying asset. The first extension is obvious, due to the discrete nature of the bounds. For the second extension we note, for instance, that the upper bound given in (4.8) for lognormally distributed jump amplitudes under constant diffusion parameters is also valid in the presence of proportional transaction costs if it is multiplied by the roundtrip transaction cost; see Proposition 1 in Constantinides and Perrakis (2002). Similar extensions may also be feasible for the lower bound of (4.11) (Proposition 5 of the same paper), although the limiting form of that result is not available and is still under study.

Appendix

Proof of Lemma 1

The proof is similar to the one used by Merton (1982), the only difference being that ε is now a bounded continuous random variable rather than a multinomial discrete one. Denote $Q_t(\delta)$ the conditional probability that $|\phi_{t+\Delta t} - \phi_t| > \delta$, given the information available at time t. Since ε is bounded, define $\bar{\varepsilon} = \max|\varepsilon| = \max(|\varepsilon_{\min}|, |\varepsilon_{\max}|)$. For any $\delta(t) > 0$, define $h(\delta)$ as the solution of the equation

$$\delta = \mu_t h + \sigma_t \bar{\varepsilon}\sqrt{h}.$$

This equation admits a positive solution

$$\sqrt{h} = \frac{-\sigma_t\bar{\varepsilon} + \sqrt{\sigma_t^2\bar{\varepsilon}^2 + 4\mu_t\delta}}{2\mu_t}.$$

For any $\Delta t < h(\delta)$ and for any possible $\phi_{t+\Delta t}$,

$$|\phi_{t+\Delta t} - \phi_t| = |\mu_t\,\Delta t + \sigma_t\varepsilon\sqrt{\Delta t}\,| < \mu_t h + \sigma_t\bar{\varepsilon}\sqrt{h} = \delta$$

so $Q_t(\delta) = \Pr(|\phi_{t+\Delta t} - \phi_t| > \delta \equiv 0$ whenever $\Delta t < h$ and hence

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} Q_t(\delta) = 0$$

The Lindeberg condition is thus satisfied. Equations (2.4) and (2.5) are satisfied by the construction of this discrete process, so the diffusion limit of (2.1) is (2.8), QED.

Proof of Lemma 2

We prove the convergence of the discretization (2.2) in the i.i.d. case[39] $\mu_t - \lambda\mu_J = \mu,\ \sigma_t = \sigma,\ J_t = J.$ Convergence in the non-i.i.d. case follows from the convergence criteria for stochastic integrals, presented in Duffie and Protter (1992). It is shown in an appendix, available from the authors on request.

The characteristic function of the terminal stock price at time *T* for a $1 initial price under the jump-diffusion process (2.9) is

$$\begin{aligned}\varphi_{JD}(\omega) &= \exp(i\omega\mu T - \frac{\omega^2\sigma^2 T}{2})\exp(-\lambda T)\sum_{N=0}^{\infty}\frac{(\lambda T)^N}{N!}[\varphi_J(\omega)]^N \\ &= \exp(i\omega\mu T - \frac{\omega^2\sigma^2 T}{2})\exp[\lambda T(\varphi_J(\omega) - 1)],\end{aligned} \tag{A.1}$$

where $\varphi_J(\omega)$ is the characteristic function of the jump distribution. The first exponential corresponds to the diffusion component and the second to the jump component.

[39] The proof is similar to that of Theorem 21.1in Jacod and Protter (2003).

The characteristic function of the discretization (2.2) is

$$\varphi(\omega) = (\lambda \Delta t \varphi_J(\omega) + 1 - \lambda \Delta t)[\exp(i\omega\mu\Delta t)\varphi_\varepsilon(\omega\sigma\sqrt{\Delta t})], \tag{A.2}$$

where $\varphi_\varepsilon(\omega)$ is the characteristic function of ε.[40] Since the distribution of ε has mean 0 and variance 1, we have

$$E[\varepsilon] = 0 = i\varphi_\varepsilon'(0)$$
$$E[\varepsilon^2] = 1 = -\varphi_\varepsilon''(0)$$

By the Taylor expansion of $\varphi_\varepsilon(\omega)$, we get

$$\varphi(\omega) = (\lambda \Delta t \varphi_J(\omega) + 1 - \lambda \Delta t)\left[\exp(i\omega\mu\Delta t)[1 - \frac{\omega^2\sigma^2\Delta t}{2} + \omega^2\sigma^2\Delta t h(\omega\sigma\sqrt{\Delta t})]\right],$$

where $h(\omega) \to 0$ as $\omega \to 0$. The multiperiod convolution has the characteristic function $\varphi(\omega)^{T/\Delta t}$. Taking the limit, we have

$$\lim_{\Delta t \to 0}[\varphi(\omega)]^{T/\Delta t} = \lim_{\Delta t \to 0} \exp\left[\frac{T}{\Delta t}\left(\ln(\lambda \Delta t \varphi_J(\omega) + 1 - \lambda \Delta t) + \ln\left[\exp(i\omega\mu\Delta t)[1 - \frac{\omega^2\sigma^2\Delta t}{2} + \omega^2\sigma^2\Delta t h(\omega\sigma\sqrt{\Delta t})]\right]\right)\right]$$
$$= \exp\left[\lambda T\left(\varphi_J(\omega) - 1\right) + i\omega\mu T - \frac{\omega^2\sigma^2 T}{2}\right] \tag{A.3}$$

after applying l'Hospital's rule. (A.3) is, however, the same as (A.1), the characteristic function of (2.9), and Levy's continuity theorem[41] proves the weak convergence of (2.2) to (2.9), QED.

Another way to characterize the limit process is its generator. Denote by z_{Dt} the diffusion component and by z_{Jt} the jump component of the return process. From equations (2.6) and (2.7), we have

$$\lim_{\Delta t \to 0} \frac{E[f(S_{t+\Delta t}, t + \Delta t)] - f(S_t, t)}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{E[f(S_t(1 + z_{D,t+\Delta t}), t + \Delta t)] - f(S_t, t)}{\Delta t} + \lambda \Delta t \frac{E[f(S_t(1 + z_{J,t+\Delta t}), t + \Delta t)] - f(S_t, t)}{\Delta t}$$
$$= (\mu_t - \lambda\mu_J) S \frac{\partial f}{\partial S} + \frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 f}{\partial S^2} + \lambda E[f(S(1 + J)) - f(S)],$$

---

[40] If, instead of (2.2) we have a mixture of the diffusion and jump components then the characteristic function becomes $\varphi(\omega) = \lambda \Delta t \varphi_J(\omega) + (1 - \lambda \Delta t)[\exp(i\omega\mu\Delta t)\varphi_\varepsilon(\omega\sigma\sqrt{\Delta t})]$. The multiperiod convolution, however, still converges to (A.3).

[41] See for instance Jacod and Protter (2003), Theorem 19.1.

which gives us the generator of the price process described by (2.9).

Proof of Lemma 3

The proof follows closely Ritchken (1985), with the important modification that the lowest stock return is bound away from bankruptcy ($z_{1t+\Delta t} > -1$). Since $Y(z_{t+\Delta t})$ is non-increasing and $Y_{1t+\Delta t} \geq Y_{2t+\Delta t} \geq ... \geq Y_{nt+\Delta t}$, we may set $Y_{nt+\Delta t} = \chi_n,\ Y_{n-1,t+\Delta t} = \chi_n + \chi_{n-1},...,Y_{1t+\Delta t} = \chi_1 + ... + \chi_n,\ \chi_j \geq 0,\ j = 1,...,n$ and replace into (3.2). We also define

$$\begin{aligned} & R\chi_j \sum_{i=1}^{i=j} p_{it+\Delta t} \equiv \overline{Y}_j \\ & \overline{c}_j = \frac{\sum_{i=1}^{j} C_{t+\Delta t}(S_t(1+z_{it+\Delta t}))p_{it+\Delta t}}{\sum_{i=1}^{j} p_{it+\Delta t}} = E\left[C_{t+\Delta t}(S_t(1+z_{t+\Delta t})) \mid z_{t+\Delta t} \leq z_{jt+\Delta t}, S_t\right] \end{aligned} \quad , \quad j = 1,...,n \tag{A.4}$$

Replacing (A.4) into (3.2) and taking into account the definitions in (3.3) we see that the LP takes now the following form:

$$\frac{1}{R}[max_{\overline{Y}_j} \sum_{j=1}^{j=n} \overline{Y}_j \overline{c}_j (\min_{\overline{Y}_j} \sum_{j=1}^{j=n} \overline{Y}_j \overline{c}_j)],$$

subject to

$$\sum_{j=1}^{j=n} \overline{Y}_j = 1, \quad \sum_{j=1}^{j=n} \overline{Y}_j (1 + \hat{z}_{jt+\Delta t}) = R, \quad \overline{Y}_j \geq 0, \quad j = 1,...,n \tag{A.5}$$

To show that the solution of (A.5) is given by the expressions in (3.4)-(3.5ab) we invoke the following property: if $C_{t+\Delta t}(S_t(1+z_{t+\Delta t}))$ is convex in $z_{t+\Delta t}$ for any given S_t then the function $\overline{c}_j(\hat{z}_{jt+\Delta t})$ is convex over the points $\hat{z}_{jt+\Delta t}, j = 1,...,n$.[42] The solution of (A.5) will first be derived graphically and then will be shown to be the optimal solution of the LP in (A.5) by using the dual formulation of the LP. The graph of $\overline{c}_j$ plotted as a function of $\hat{z}_{jt}$ is shown in Figure A.1. The first and last constraints indicate that all admissible solutions of (A.5) must lie within the convex hull of the points on the graph. Further, the second constraint dictates that it should lie on the vertical line emanating from the point $R-1$ on the horizontal axis, as shown in Figure A.1.

The optimal solution of (A.5) and, hence, of (3.2) can be easily visualized from Figure A.1. For instance, the upper bound is equal to

[42] See Ritchken (1985, p. 1227).

$$\frac{1}{R}[\overline{Y}_1\overline{c}_1+\overline{Y}_n\overline{c}_n]=\frac{1}{R}[\overline{Y}_1 C_{t+\Delta t}(S_t(1+z_{1t+\Delta t}))+\overline{Y}_n E^P[C_{t+\Delta t}(S_t(1+z_{t+\Delta t}))\big|S_t]]\,. \tag{A.6}$$

Replacing into the constraints of (A.5) and solving we find that the optimal solution of the LP is equal to

$$\frac{1}{R}\left[\frac{R-1-\hat{z}_{1t+\Delta t}}{\hat{z}-\hat{z}_{1t+\Delta t}}\overline{c}_n+\frac{\hat{z}+1-R}{\hat{z}-\hat{z}_{1t+\Delta t}}\overline{c}_1\right]=\frac{1}{R}\left[\frac{R(\overline{c}_n-\overline{c}_1)}{\hat{z}-\hat{z}_{1t+\Delta t}}+\frac{(\hat{z}+1)\overline{c}_1-(1+\hat{z}_{1t+\Delta t})\overline{c}_n}{\hat{z}-\hat{z}_{1t+\Delta t}}\right]. \tag{A.7}$$

To show that this is, indeed, the solution of the maximization problem in (A.5) we consider its dual

$$\frac{1}{R}[\min_{u_1,u_2}(u_1+u_2R)]$$

subject to

$$u_1+u_2(1+\hat{z}_{jt+\Delta t})\geq\overline{c}_j,\;\; u_1,u_2 \;\text{ unrestricted}\,,\quad j=1,\ldots,n\,. \tag{A.5)'$$

It suffices to show that the solution $u_1=\dfrac{(\hat{z}+1)\overline{c}_1-(1+\hat{z}_{1t+\Delta t})\overline{c}_n}{\hat{z}-\hat{z}_{1t+\Delta t}}$, $u_2=\dfrac{\overline{c}_n-\overline{c}_1}{\hat{z}-\hat{z}_{1t+\Delta t}}$, which is feasible in the primal and is conjectured to be the optimal objective function in (A.7), is also feasible in the dual. Replacing this solution into the constraints of (A.5)'and invoking the convexity of $\overline{c}_j(\hat{z}_{jt+\Delta t})$ we find that the inequalities $u_1+u_2(1+\hat{z}_{jt+\Delta t})\geq\overline{c}_j$ are satisfied for all $j=1,\ldots,n$. Hence, (A.7) is the upper bound in (A.5), and it can be easily seen that this upper bound value corresponds to $\frac{1}{R}E^U[C_{t+\Delta t}(S_t(1+z_t))\big|S_t]$ as in (3.4)-(3.5a), QED. An identical proof also holds for the lower bound (3.4)-(3.5b). The proof can be extended without reformulation to the case where the distribution $P(z_{t+\Delta t}\big|S_t)$ is continuous, by discretizing the distribution into equally spaced intervals $\Delta z_{t+\Delta t}$, applying the relations (3.4)-(3.5) and then letting $\Delta z_{t+\Delta t}$ tend to zero, in which case the bounds are given by (3.4)-(3.6).[43]

(Figure A.1 about here)

---

[43] We may impose the requirement that the pricing kernel be continuously decreasing in $z_{t+\Delta t}$ by replacing in (A.5) the nonnegativity constraints by $\overline{Y}_j\geq\delta>0,\; j=1,\ldots,n$, where δ is a parameter. It can then be easily shown that the LP would yield upper and lower bounds that are, respectively, decreasing and increasing functions of δ, becoming for $\delta=0$ equal to the bounds in (3.4)-(3.5ab); the latter are thus the limits of the option bounds for strictly decreasing pricing kernels.

Proof of Proposition 2

Under the upper bound probability given by (3.5a) or (3.6), the returns process becomes

$$\bar{z}_{t+\Delta t} = \begin{cases} z_{t+\Delta t} & \text{with probability} \quad 1-Q \\ z_{\min,t+\Delta t} & \text{with probability} \quad Q \end{cases},$$

where Q is the following probability

$$\begin{aligned} Q &= \frac{\hat{z} - r\,\Delta t}{(\hat{z} - z_{\min,t+\Delta t})} \\ &= \frac{\mu_t\,\Delta t - r\,\Delta t}{\mu_t\,\Delta t - (\mu_t\,\Delta t + \sigma_t \varepsilon_{\min}\sqrt{\Delta t})} = -\frac{\mu_t - r}{\sigma_t \varepsilon_{\min}}\sqrt{\Delta t} \end{aligned}$$

From the definition of $z_{t+\Delta t}$ given in (2.1) we get

$$\bar{z}_{t+\Delta t} = \mu(S_t,t)\Delta t + \sigma(S_t,t)\sqrt{\Delta t}\begin{cases} \varepsilon & \text{with probability} \quad 1-Q \\ \varepsilon_{\min} & \text{with probability} \quad Q \end{cases} \tag{A.8}$$

The random component of the returns in (A.8) has a bounded discrete or continuous distribution, so by Lemma 1 the upper bound process satisfies the Lindeberg condition. The upper bound distribution (A.8) has the mean

$$\begin{aligned} E^{U_t}[\bar{z}_{t+\Delta t}] &= \mu_t \Delta t + (1 + \frac{\mu_t - r}{\sigma_t \varepsilon_{\min}}\sqrt{\Delta t})(\sigma_t\sqrt{\Delta t})E[\varepsilon] \\ &\quad - \frac{\mu_t - r}{\sigma_t \varepsilon_{\min}}\sqrt{\Delta t}(\sigma_t\sqrt{\Delta t})\varepsilon_{\min} \\ &= \mu_t \Delta t - \frac{\mu_t - r}{\sigma_t \varepsilon_{\min}}\sqrt{\Delta t}(\sigma_t\sqrt{\Delta t})\varepsilon_{\min} = r\Delta t \end{aligned}$$

Its variance is

$$\begin{aligned} Var^{U_t}[\bar{z}_{t+\Delta t}] &= \sigma_t^2 \Delta t\left[\left(1 + \frac{\mu_t - r}{\sigma_t \varepsilon_{\min}}\sqrt{\Delta t}\right)Var[\varepsilon] \right. \\ &\quad \left. - \frac{\mu_t - r}{\sigma_t \varepsilon_{\min}}\sqrt{\Delta t}\,\varepsilon_{\min}^2\right] + o(\Delta t) \\ &= \sigma_t^2 \Delta t\left[1 + \frac{\mu_t - r}{\sigma_t \varepsilon_{\min}}\sqrt{\Delta t} - \frac{\mu_t - r}{\sigma_t \varepsilon_{\min}}\sqrt{\Delta t}\,\varepsilon_{\min}^2\right] + o(\Delta t) \\ &= \sigma_t^2 \Delta t + o(\Delta t) \end{aligned}$$

Consequently, the upper bound process converges weakly to the diffusion (4.1), QED.

We prove the convergence of the lower bound for the case given by (3.6), under a continuous probability distribution D of ε. A different proof applies to the discrete case given by (3.5b), which is based on the fact that for a sufficiently low Δt we have $\hat{z}_{n-1,t+\Delta t} < R-1 < \hat{z}_{n,t+\Delta t} = \hat{z}_{t+\Delta t}$ in (3.5b); it is in an appendix available from the authors on request. The transformed returns process becomes

$$\underline{z}_{t+\Delta t} = \mu(S_t, t)\Delta t + \sigma(S_t, t)\hat{\varepsilon}_{t+\Delta t}\sqrt{\Delta t},$$

where $\hat{\varepsilon}_{t+\Delta t}$ is a truncated random variable $\{\hat{\varepsilon}_{t+\Delta t} \mid \varepsilon \le \overline{\varepsilon}_t\}$, with $\overline{\varepsilon}_t$ found from the condition $E^{L_t}[\underline{z}_{t+\Delta t}] = r\Delta t$. Since $\hat{\varepsilon}_{t+\Delta t}$ is truncated from a bounded distribution the Lindeberg condition is satisfied. The risk neutrality of the lower bound distribution implies that

$$\mu_t \Delta t + \sigma_t \sqrt{\Delta t} E[\hat{\varepsilon}_{t+\Delta t}] = r\Delta t,$$

and the mean of $\hat{\varepsilon}_{t+\Delta t}$ is

$$E[\hat{\varepsilon}_{t+\Delta t}] = -\frac{\mu_t - r}{\sigma_t}\sqrt{\Delta t} \tag{A.9}$$

Since this random variable is drawn from a distribution that is truncated from the distribution D of ε we get

$$E[\hat{\varepsilon}_{t+\Delta t}] = \frac{1}{\Pr(\varepsilon_t < \overline{\varepsilon}_t)}\int_{\varepsilon_{\min}}^{\overline{\varepsilon}_t} \varepsilon dD(\varepsilon) = \frac{1}{D(\overline{\varepsilon}_t)}\int_{\varepsilon_{\min}}^{\overline{\varepsilon}_t} \varepsilon dD(\varepsilon) = -\frac{\mu_t - r}{\sigma_t}\sqrt{\Delta t} \tag{A.10}$$

From (A.10) we can easily see that

$$\frac{dE[\hat{\varepsilon}_{t+\Delta t}]}{d(\Delta t)} = \frac{dE[\hat{\varepsilon}_{t+\Delta t}]}{d\overline{\varepsilon}_t}\frac{d\overline{\varepsilon}_t}{d(\Delta t)} < 0\,. \tag{A.11}$$

Since the first term in the product is clearly positive, it follows that $\frac{d\overline{\varepsilon}_t}{d(\Delta t)} < 0$. For every Δt, therefore, there exists a value $\overline{\varepsilon}_t(\Delta t)$ solving (A.10), which is a decreasing function of Δt. By assumption we have $E[\varepsilon] = 0$, implying that

$$\int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \varepsilon dD(\varepsilon) = \int_{\varepsilon_{\min}}^{\overline{\varepsilon}_t} \varepsilon dD(\varepsilon) + \int_{\overline{\varepsilon}_t}^{\varepsilon_{\max}} \varepsilon dD(\varepsilon) = 0, \tag{A.12}$$

with $\varepsilon_{\max} > 0$. Since $\frac{d\overline{\varepsilon}_t}{d(\Delta t)} < 0$ from (A.11), there exists a value $\Delta t = \delta$ such that $0 \le \overline{\varepsilon}_t(\delta) \le \overline{\varepsilon}_t \le \varepsilon_{\max}$, for any $\Delta t < \delta$. From (A.9)-(A.12) ,we get

$$
\begin{aligned}
\frac{\mu_t - r}{\sigma_t}\sqrt{\Delta t} &= \frac{1}{\Pr(\varepsilon < \overline{\varepsilon}_t)}\int_{\overline{\varepsilon}_t}^{\varepsilon_{\max}} \varepsilon dD_D(\varepsilon) \\
&\geq \frac{1}{1-\Pr(\varepsilon > \overline{\varepsilon}_t)}\int_{\overline{\varepsilon}_t}^{\varepsilon_{\max}} \overline{\varepsilon}_t dD_D(\varepsilon) \qquad \text{(A.13)} \\
&= \frac{\overline{\varepsilon}_t \Pr(\varepsilon > \overline{\varepsilon}_t)}{1-\Pr(\varepsilon > \overline{\varepsilon}_t)} > \frac{\overline{\varepsilon}_t(\delta)\Pr(\varepsilon > \overline{\varepsilon}_t)}{1-\Pr(\varepsilon > \overline{\varepsilon}_t)}.
\end{aligned}
$$

From the last inequality of (A.13) we get

$$
\Pr(\varepsilon > \overline{\varepsilon}_t) \leq \frac{\frac{\mu_t - r}{\sigma_t}\sqrt{\Delta t}}{\overline{\varepsilon}_t(\delta) + \frac{\mu_t - r}{\sigma_t}\sqrt{\Delta t}} = O(\sqrt{\Delta t}). \qquad \text{(A.14)}
$$

(A.14) implies that as $\Delta t \to 0$ the probability that $\varepsilon > \overline{\varepsilon}_t$ tends to zero. Therefore, the limit lower bound distribution contains all the possible outcomes of ε . This result is used to compute the limit of the variance of $\hat{\varepsilon}_{t+\Delta t}$

$$
\begin{aligned}
\lim_{\Delta t \to 0} Var(\hat{\varepsilon}_{t+\Delta t}) &= \lim_{\Delta t \to 0}\{E[\hat{\varepsilon}^2_{t+\Delta t}] - (E[\hat{\varepsilon}_{t+\Delta t}])^2\} \\
&= \lim_{\Delta t \to 0}\{\frac{1}{\Pr(\varepsilon < \overline{\varepsilon}_t)}\int_{\varepsilon_{\min}}^{\overline{\varepsilon}_t} \varepsilon^2 dD(\varepsilon) - \left(\frac{\mu_t - r}{\sigma_t}\right)^2 \Delta t\} \qquad \text{(A.15)} \\
&= \lim_{\Delta t \to 0}\{\int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \varepsilon^2 dD(\varepsilon) - \left(\frac{\mu_t - r}{\sigma_t}\right)^2 \Delta t\} = 1,
\end{aligned}
$$

where the third equality in (A.15) applies the conclusion derived from (A.14) and the last equality uses the fact that

$$
Var(\varepsilon) = \int_{\varepsilon_{\min}}^{\varepsilon_{\max}} \varepsilon^2 dD(\varepsilon) = 1
$$

It follows that

$$
Var^{L_t}[\underline{z}_{t+\Delta t}] = \sigma_t^2 \Delta t + O(\Delta t)^2
$$

The diffusion limit is, therefore, the process described by equation (4.1), QED.

Proof of Proposition 3

As with Proposition 2, we consider the multiperiod discrete time bounds of section 3, obtained by successive expectations under the risk-neutral upper bound distribution. We then seek the limit of this distribution as $\Delta t \to 0$. Ignoring terms $o(\Delta t)$, the probability Q used in equation (3.6) is given by

$$
Q = \frac{E(z_{t+\Delta t}) - r\Delta t}{E(z_{t+\Delta t}) - J_{\min}} = \frac{(\mu_t - r)\Delta t}{\mu_t \Delta t - \sigma \max(|\varepsilon|)\sqrt{\Delta t} - J_{\min}} = \lambda_{Ut}\Delta t,
$$

where

$$\lambda_{Ut} = -\frac{\mu_t - r}{J_{\min}},$$

Observe that λ_{Ut} is always positive since $J_{\min} < 0$ and $E(z_{t+\Delta t}) > r\Delta t$. Hence, the discrete time upper bound process is, by (2.2) and (3.6)

$$z_{t+\Delta t} = \begin{cases} z_{D,t+\Delta t} + J\Delta N & \text{with probability} \quad 1-\lambda_{Ut}\Delta t \\ z_{D,t+\Delta t} + J_{\min} & \text{with probability} \quad \lambda_{Ut}\Delta t \end{cases},$$

The outcomes of this process and their probabilities are as follows:

$$z_{t,t+\Delta t} = \begin{cases} z_{D,t+\Delta t} & \text{with probability} \quad (1-\lambda\Delta t)(1-\lambda_{Ut}\Delta t), \\ z_{D,t+\Delta t} + J & \text{with probability} \quad \lambda\Delta t(1-\lambda_{Ut}\Delta t), \\ z_{D,t+\Delta t} + J_{\min} & \text{with probability} \quad \lambda_{Ut}\Delta t. \end{cases}$$

By removing the terms in $o(\Delta t)$, the upper bound process outcomes become

$$z_{t+\Delta t} = \begin{cases} z_{D,t+\Delta t} & \text{with probability} \quad 1-(\lambda+\lambda_{Ut})\Delta t \\ z_{D,t+\Delta t} + J_t^U & \text{with probability} \quad (\lambda+\lambda_{Ut})\Delta t \end{cases}, \tag{A.16}$$

where J_t^U is given by (4.4). This process, however, corresponds to (4.2), QED.

The generator of the price process, which is also reflected in equation (4.5), is

$$\begin{aligned} \mathcal{A}^U f &= \left[r - (\lambda+\lambda_{Ut})\mu_J^U \right] S\frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} \\ &\quad + \frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 f}{\partial S^2} + (\lambda+\lambda_{Ut})E^U[f(S(1+J_t^U)) - f(S)]. \end{aligned} \tag{A.17}$$

Proof of Proposition 4

The proof is very similar to those of Lemma 2 and Proposition 3. Assuming, for simplicity, that both ε and J have continuous distributions, we may apply equation (3.6). From the proof of Lemma 1, it is clear that as $\Delta t \to 0$ all the outcomes of the diffusion component will be lower than $\bar{J}_t$. Therefore, the limiting distribution will include the whole diffusion component and a truncated jump component. The maximum jump outcome in this truncated distribution is obtained from the condition that the distribution is risk neutral, which is expressed in (4.10). We observe that the lower bound distribution over $(t, t+\Delta t)$ is the sum of the diffusion component and a jump of intensity λ and log-amplitude distribution J_t^L, the truncated distribution $\{J \mid J \le \bar{J}_t\}$.

$$z_{t+\Delta t} = \begin{cases} z_{D,t+\Delta t} & \text{with probability} \quad 1-\lambda\Delta t \\ z_{D,t+\Delta t} + J_t^L\Delta N, & \text{with probability} \quad \lambda\Delta t \end{cases}.$$

By Lemma 2 this process converges weakly for $\Delta t \to 0$ to the jump-diffusion process (4.9), QED. The generator of the price process is

$$\begin{aligned}\mathcal{A}^L f &= \left[r - \mu_{Jt}^L\right] S \frac{\partial f}{\partial S} + \frac{\partial f}{\partial t} \\ &+ \frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 f}{\partial S^2} + \lambda E^L[f(S(1+J_t^L)) - f(S)].\end{aligned} \tag{A.18}$$

which appears in equation (4.11).

Proof of Proposition 5

Since Lemma 4 holds and $C_t(S_t)$ is convex, Lemma 5 holds as well, and the option price lies within the set of bounds given by (5.3). We then consider the limit of these bounds as $\Delta t \to 0$, as in the proof of Proposition 2.

Assume for simplicity that the distributions of both ε and η are continuous. It then follows from (5.3) and (5.5) that the two bounds are given by recursive expectations of the option payoff, taken with respect to the following distributions, which replace (3.6), with U_t and L_t the same distributions as those given by (3.6).

$$\hat{U}(v_{t+\Delta t}) = D_v(\eta)U_t(z_{t+\Delta t}), \;\; \hat{L}(v_{t+\Delta t}) = D_v(\eta)L_t(z_{t+\Delta t})\,. \tag{A.19}$$

The rest of the proof follows along the lines of the proof of Proposition 2. It is easy to show that $E^{\hat{U}_t}[v_{t+\Delta t}] = E^{\hat{L}_t}[v_{t+\Delta t}] = r\Delta t$, taking into account (5.5) and (5.10). Further, $Var^{\hat{U}_t}[v_{t+\Delta t}] = \sigma_t^{v2}(1-\rho^2)\Delta t + Var^{U_t}(\bar{v}(z_{t+\Delta t})),\; Var^{\hat{l}_t}[v_{t+\Delta t}] = \sigma_t^{v2}(1-\rho^2)\Delta t + Var^{L_t}(\bar{v}(z_{t+\Delta t}))$ implying that $Var^{\hat{U}_t}[v_{t+\Delta t}] = Var^{\hat{l}_t}[v_{t+\Delta t}] = \sigma_t^{v2}\Delta t + o(\Delta t)$ as in the proof of Proposition 2, thus completing the proof.

Proof of Proposition 6

Since Lemma 5 obviously holds, the option bounds are found from the limits of relations (5.3) as $\Delta t \to 0$. Note that the distributions U_t, L_t in (5.3) apply to the conditional expectations of the options given $z_{t+\Delta t}$, or given $z_{D,t+\Delta t}$ and $z_{J,t+\Delta t}$. The conditional expectations of the stock returns form the mixture

$$\bar{v}(z_{t+\Delta t}) = \begin{cases} \bar{v}(z_{D,t+\Delta t}) & \text{with probability} \quad 1-\lambda\Delta t \\ J_{SI} & \text{with probability} \quad \lambda\Delta t \end{cases}. \tag{A.20}$$

For the upper bound, as with the proof of Proposition 3 and ignoring terms $o(\Delta t)$, the probability Q used in equation (3.6) is given by

$$Q = \frac{E(\bar{v}(z_{t+\Delta t})) - r\Delta t}{E(\bar{v}(z_{t+\Delta t})) - J_{SI\min}} = \frac{(m_t - r)\Delta t}{m_t \Delta t - \sigma_t^v \max(|\varepsilon|)\sqrt{\Delta t} - J_{SI\min}} = \lambda_{Ut}\Delta t,$$

and the discrete time upper bound process is, by (5.11)-(5.12) and (3.6),

$$v_{t+\Delta t} = \begin{cases} v_{D,t+\Delta t} + J_S \Delta N & \text{with probability} \quad 1 - \lambda_{Ut}\Delta t \\ v_{D,t+\Delta t} + J_{SI\min} & \text{with probability} \quad \lambda_{Ut}\Delta t \end{cases}.$$

As with the proof of Proposition 3, by removing the terms in $o(\Delta t)$ this process becomes

$$v_{t+\Delta t} = \begin{cases} v_{D,t+\Delta t} & \text{with probability} \quad 1 - (\lambda + \lambda_{Ut})\Delta t \\ v_{D,t+\Delta t} + J_{tS}^U & \text{with probability} \quad (\lambda + \lambda_{Ut})\Delta t \end{cases},$$

where J_{tS}^U is given by (5.15). For $\Delta t \to 0$ this process becomes (5.15), QED. A similar proof holds for the lower bound, virtually identical to the proof of Proposition 4. Since for $\Delta t \to 0$ all outcomes of the diffusion component in the conditional distribution of $v_{t+\Delta t}$ will be lower than the jump component corresponding to $\overline{\mathrm{X}}^{\mathrm{t}}$. Hence, the lower bound distribution over $(t, t+\Delta t)$ is the sum of the diffusion component and a jump of intensity λ and log-amplitude distribution J_{tS}^L, the truncated distribution $\{J_S \,|\, \mathrm{X} \le \overline{\mathrm{X}}^t\}$. This, however, tends to (5.16) for $\Delta t \to 0$, QED.

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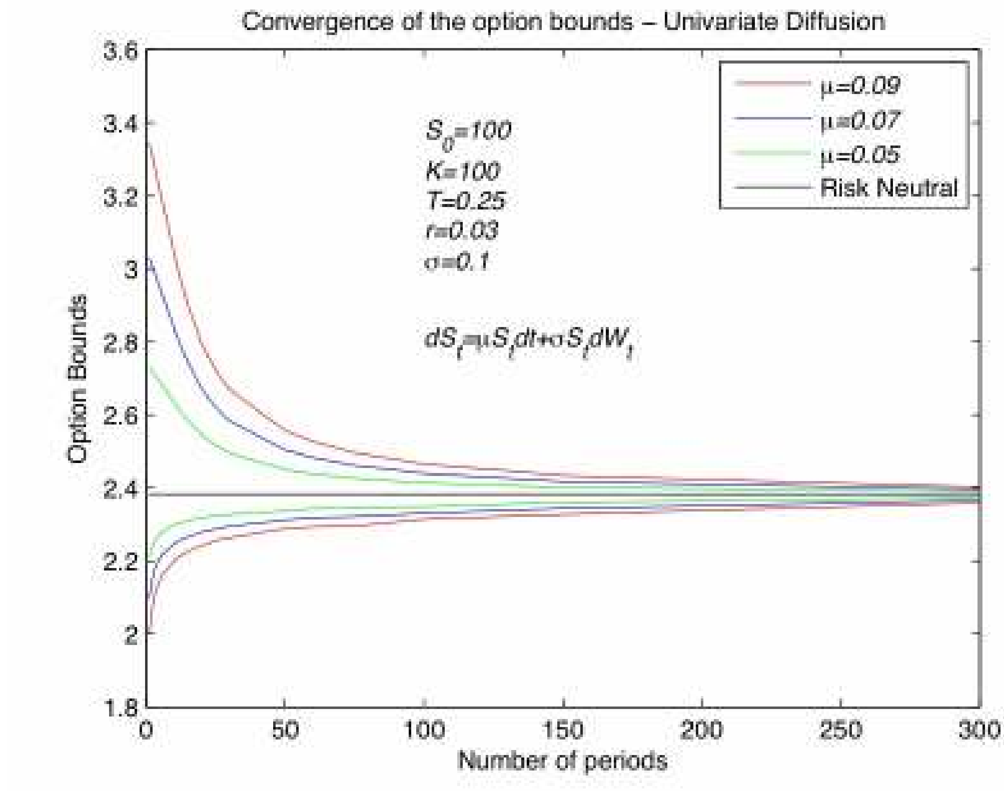
Table 4.1

Number of periods	μ=0.09		μ=0.07		μ=0.05		Risk Neutral
	Upper	Lower	Upper	Lower	Upper	Lower	
1	3.4606	2.0066	3.1376	2.1014	2.8309	2.2096	2.4107
2	3.4599	2.0781	3.1372	2.1552	2.8307	2.2424	2.4107
3	3.4596	2.1194	3.1369	2.1867	2.8305	2.2627	2.4107
4	3.4578	2.1461	3.1357	2.2073	2.8299	2.2765	2.4107
5	3.4542	2.1642	3.1334	2.2225	2.8286	2.2864	2.4107
6	3.45	2.1809	3.1307	2.2335	2.8272	2.2937	2.4107
10	3.4303	2.2187	3.1182	2.2641	2.8204	2.3142	2.4107
12	3.4197	2.231	3.1115	2.2735	2.8168	2.3196	2.4107
15	3.4036	2.2463	3.101	2.2846	2.8112	2.3277	2.4107
20	3.3764	2.2618	3.0832	2.2968	2.8018	2.3364	2.4107
25	3.3491	2.276	3.0654	2.307	2.7923	2.3419	2.4107
30	3.3217	2.2829	3.0474	2.3148	2.7823	2.3464	2.4107
50	3.2143	2.3091	2.9766	2.3328	2.7433	2.3562	2.4107
60	3.1619	2.3141	2.942	2.3387	2.7242	2.3629	2.4107
75	3.0862	2.3165	2.8915	2.3419	2.6961	2.367	2.4107
100	2.9758	2.3362	2.8162	2.3515	2.6534	2.3682	2.4107
150	2.823	2.3486	2.7084	2.3652	2.5901	2.3814	2.4107
300	2.6311	2.3817	2.5695	2.3882	2.5059	2.395	2.4107

Table 4.2

Strike	Upper Bound	Lower Bound	Risk Neutral
95	6.1155	6.0023	6.0119
96	5.2916	5.1575	5.1701
97	4.518	4.365	4.3808
98	3.8025	3.6345	3.6531
99	3.1514	2.9736	2.9946
100	2.5695	2.3882	2.4107
101	2.0594	1.8811	1.904
102	1.6213	1.4521	1.4743
103	1.2529	1.0977	1.1184
104	0.9499	0.8122	0.8306
105	0.7062	0.588	0.6036



Convergence of the option bounds – Univariate Diffusion
μ=0.09
μ=0.07
μ=0.05
Risk Neutral
S0=100
K=100
T=0.25
r=0.03
σ=0.1
dSt=μStdt+σStdWt
Option Bounds
Number of periods
3.6
3.4
3.2
3
2.8
2.6
2.4
2.2
2
1.8
0
50
100
150
200
250
300

Figure 4.1

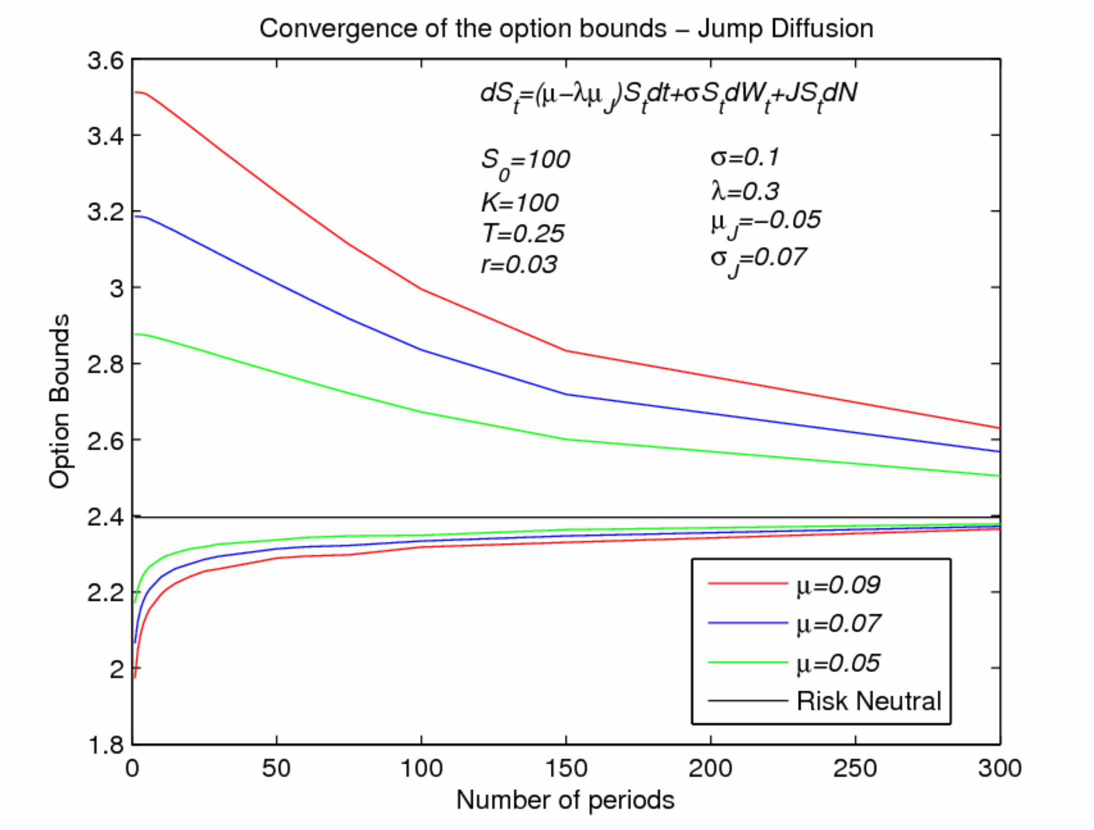


Figure 4.2

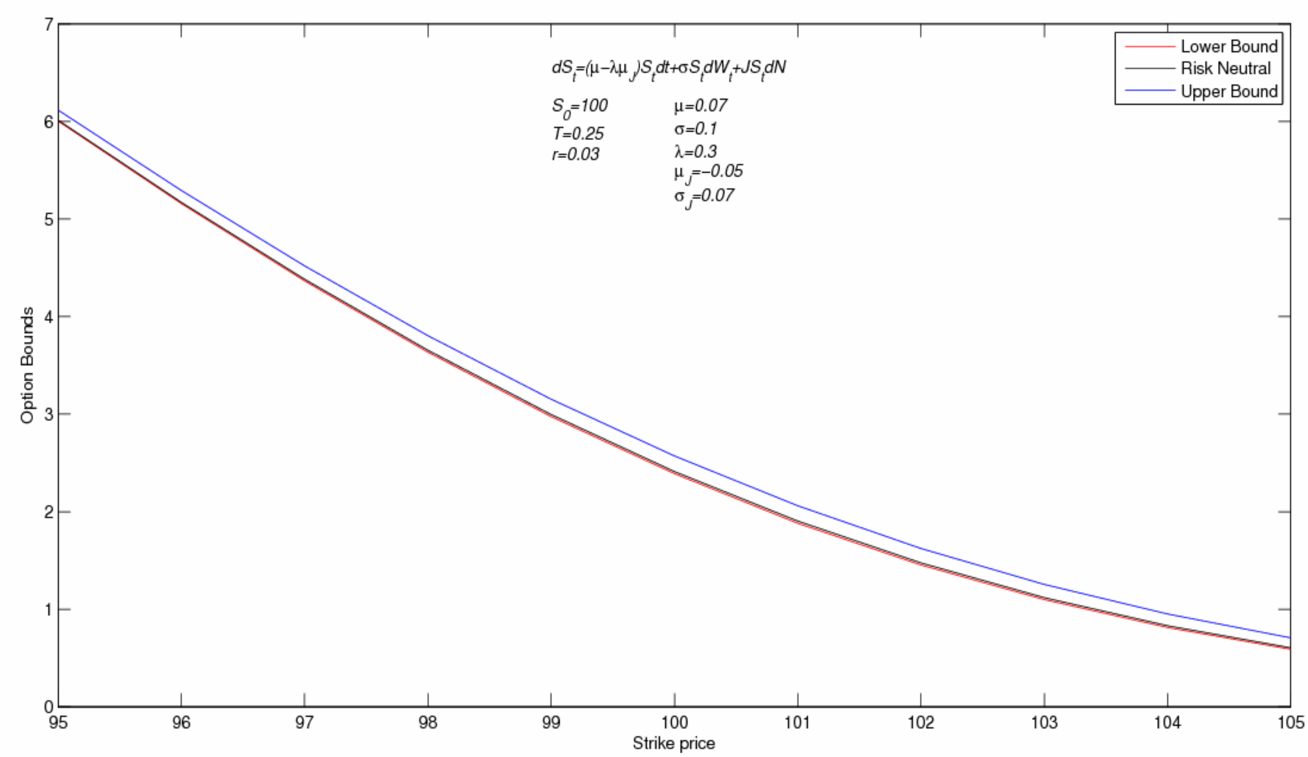


Figure 4.3

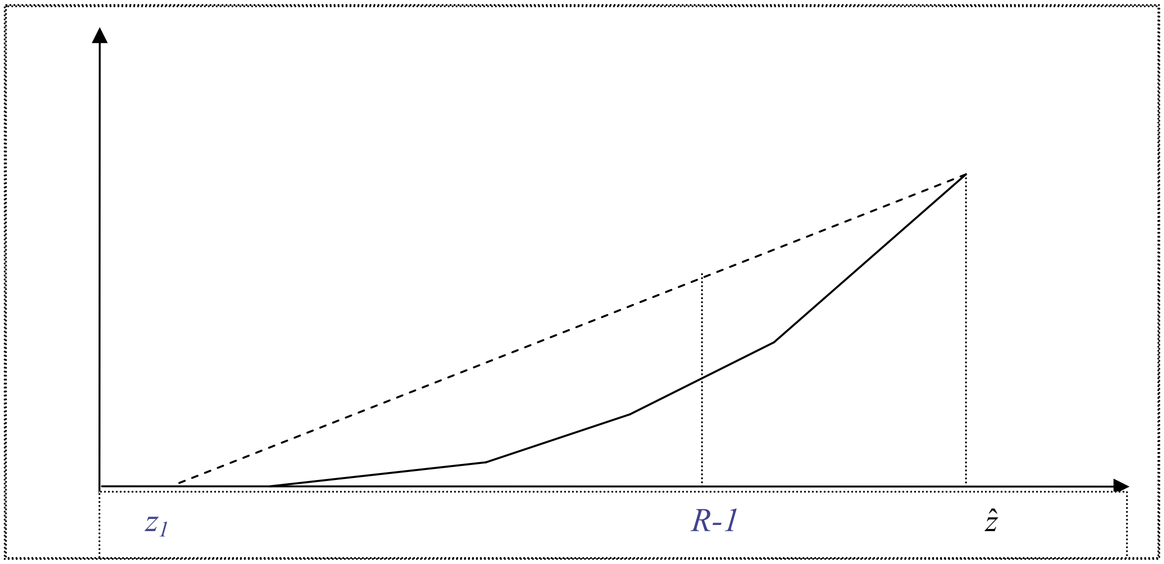


Figure A.1