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Free vibration of thin rectangular steel plates with geometrically-nonlinear load-displacement behavior

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ABSTRACT

This paper provides means for obtaining the first three *significant* vibration modes for rectangular plates based on mass participation ratios. A non-dimensional frequency parameter is presented which results into the vibration frequency of rectangular plates at each of these three significant modes. Various aspect ratios and four combinations of boundary conditions at the plate edges are studied. A correlation between the nonlinear load-deformation behavior of the plate and its vibrational behavior is also presented accordingly. It is demonstrated that the vibration frequency of the studied rectangular plates increases significantly upon increasing the applied lateral pressure if the large deformation effects are considered in the analysis. The easy-to-follow method of frequency calculation presented in this paper is useful for assessing the dynamic characteristics of rectangular plates with or without lateral pressure that are subject to vibration.

1. Introduction

Stiffened plates with rectangular panels are one of the most common forms of thin walled structures in industrial, naval and aerospace structures. The walls of industrial ducts, ship hulls, rectangular bins and aircraft wings are examples of such plates. The vibration of rectangular plates has gained the attention of many researchers in the past decades. Structural vibration analysis involves studying the vibration properties of any structure that is subjected to any form of vibrating force. Resonance conditions where the forcing frequency and the natural frequency of the structure are the same (or very close) are to be avoided. Although there are no code regulations about the resonance checking, structural engineers tend to set the limits of $\pm 20\%$ as the resonance domain, i.e. if the forcing frequency is within the range of 0.8–1.2 times the natural frequency, the structure is considered as “prone to resonance”. However, there are still questions to be answered e.g.: “How many modes should be considered?” and “Which modes are important?” that are involved with the vibration analysis of structures. Answering these questions for the cases of single-degree-of-freedom (SDOF) and multi-degrees-of-freedom (MDOF) structures involves the measurement of the participation of the structural mass in the desired DOFs. For the case of SDOF systems, there is only one mode of vibration and the mass participation for that mode is merely 100%. The vibration frequency at this mode is often called the “fundamental frequency”. Similarly, The fundamental frequency in MDOF systems corresponds to the mode with the highest mass participation which is usually achieved

at the first degree of freedom.

While the main aspects of the structural design of rectangular plates are the load-deformation and load-stress behavior, vibration-induced limit states such as fatigue and loosening of connections might cause unforeseen deficiencies during the service life of the structure if it is prone to resonance with the applied dynamic loads.

Several studies in terms of closed-form solutions and numerical methodologies to calculate the frequencies of vibration of rectangular plates are available in the literature including Gorman [13], Soedel [35], Amabili [2] and Chakraverty [6]. Effects of several parameters on the natural frequency of rectangular plates are the topic of further research including the studies by Liew et al. [19] wherein the effects of in-plane isotropic pressure on the vibration response of thick rectangular plates were discussed. More recently, Xiang et al. [43] presented one of the first known exact solutions for the vibration of rectangular multi-span plates with two opposite edges simply supported using the Levy type solution method and the state-space technique. Phillips and Jubb [27] presented a series of tests to verify the natural frequency of vibration of an approximately clamped rectangular plate due to increasing lateral distortion against the existing theory initially presented by Reissner [31]. The initial distortion of the plate in their experimental tests was applied by deflecting the plate using a rigid block at the center of the plate and a hydraulic jack prior to the vibration test. Manzanares-Martínez et al. [22] presented the lower normal modes of vibration of rectangular plates through experimental and theoretical analysis. Their experimental tests involved employing electromagnetic-acoustic

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Nomenclature

B	breadth of rectangular plate
c	viscous damping constant
D	flexural rigidity of plate material
E	modulus of elasticity of plate material
f	external excitation as a function of time
h	thickness of plate
L	length of rectangular plate
N_x	membrane force in x direction
N_y	membrane force in y direction
N_{xy}	shear force in xy plane
q	lateral pressure applied to the plate surface

q^*	non-dimensional lateral pressure
S	plate aspect ratio
Δ	maximum deflection at the center of rectangular plate
ω^*	non-dimensional frequency parameter
ω	vibration frequency of plate
ν	Poisson's ratio of plate material
ρ	mass density per unit volume of material
$w(x,y,t)$	dynamic displacement of a given point on plate surface (x,y) at specific time (t)
$\dot{w}$	velocity of any given point on plate surface (x,y) at specific time
$\ddot{w}$	acceleration of any given point on plate surface (x,y) at specific time

transducers on a thin rectangular plate in order to measure the frequencies of vibration mainly in the acoustic modes with high frequencies.

Cho et al. [8] studied the vibration of rectangular plates for the case of bottom plate of fluid containers where the plate is in direct contact with fluid. The fluid-structure interaction was the focus of that study using the Lagrange's equation of motion to relate the potential and kinetic energies of the plate structure and surrounding fluid kinetic energy. The study of the effects of imperfections on the vibration of rectangular plates was further extended by Zeng et al. [45] where a loaded side-cracked plate was studied using the Moving-Least-Square (MLS-Ritz) method. Huang and Lin [14] also studied the vibration of rectangular plates with a straight-through crack using Fourier cosine series with domain decomposition. Lately, Wang et al. [41] studied the effects of rectangular openings on the vibration characteristics of rectangular plates in order to establish a methodology for noise reduction and vibration control in thin-walled structures with openings using Fourier series.

It has to be mentioned that the concept of *nonlinear vibration of plates* that includes the response of plates to periodic loading with large amplitudes is out of the scope of this paper. This paper mainly focuses on the free vibration and fundamental frequencies of rectangular steel plates with various aspect ratios and the effects of large deformations caused by static lateral pressure on those frequencies.

2. Theoretical background, FE modeling and verification

The earliest attempts on the vibration analysis of rectangular plates in presence of in-plane (membrane and shear) forces were made by Dawe [9] and followed by alternative solutions including numerical methods by Mei and Yang [23], Dickinson [10], and Bassily and Dickinson [4]. Chan and Foo [7] were amongst the first to apply the *Finite Strip method* to solve the case of rectangular plate vibration with membrane forces.

The general equation of motion for a rectangular plate under lateral and in-plane loading is presented in Eq. (1).

$$D \left[\left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 \right] + chw + \rho h \ddot{w} = f + q + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} \quad (1)$$

In which D is the flexural rigidity of the plate calculated as $D = Eh^3 / 12(1 - \nu^2)$, c is the viscous damping, ρ is the mass density per unit volume of plate material, h is the plate thickness and $w(x, y, t)$ is the dynamic displacement at any given point of plate surface at specific time (t). The first and second derivatives of w with respect to t (i.e. $\dot{w}$ and $\ddot{w}$) represent the velocity and acceleration at a specific time. On the right side of Eq. (1), f is the external excitation as a function of time, q is the lateral pressure applied at the plate surface while N_x and N_y are the

membrane forces distributed along the edges of plate and N_{xy} is the in-plane shear force, respectively.

The simplest form of plate vibration analysis represents the free vibration of unloaded rectangular plates with negligible damping. In this case, the forcing terms at the right side and the viscous damping term at the left side of Eq. (1) are set to zero and Eq. (1) reduces into Eq. (2).

$$D \left[\left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 \right] + \rho h \ddot{w} = 0 \quad (2)$$

There is an infinite number of natural frequencies (and mode shapes) available from the closed form or numerical solutions to Eq. (2) for various boundary conditions of rectangular plates [2]. As an example, the closed form solution to Eq. (2) for rectangular plates with all edges simply supported could be presented as Eq. (3).

$$\omega_{m,n} = \pi^2 [(m/L)^2 + (n/B)^2] \sqrt{D/\rho h} \quad (3)$$

In which m and n indicate the number of wave fronts along the length (L) and breadth (B) of the rectangular plate for any particular mode shape with natural frequency $\omega_{m,n}$. Amabili [2] verified the natural frequencies of 15 modes of vibration resulted from Eq. (3) experimentally for the case of a rectangular aluminium plate.

In another case, the free vibration of rectangular plates with negligible damping in presence of membrane forces was studied by Singh and Dey [34] using the energy method. They discretized the total energy of free vibration of the system by replacing the derivative terms with their Finite Difference equivalents and used a specific energy minimization technique to solve the resulting eigenvalue problem. One step further into the case of rectangular plate vibration with membrane forces, Leissa and Kang [17] presented one of the first exact solutions for the case of vibration and buckling of rectangular plates with a particular set of boundary conditions in which two opposite edges of the plate were clamped and the two other edges were simply supported. A linearly variable in-plane load was considered to act at the simply supported edges [17]. A similar case was presented by Wang et al. [40] that simplified Eq. (1) into the form presented in Eq. (4) for the case of free vibration of rectangular plates with membrane forces acting at opposite edges along x axis [40].

$$D \left[\left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 \right] + \rho h \ddot{w} = N_x \frac{\partial^2 w}{\partial x^2} \quad (4)$$

Wang et al. [40] used the differential quadrature method to solve this equation and calculated the vibration frequency for six mode shapes of the rectangular plate ($m = 1-3$, $n = 1$ and 2). They estimated the $w(x,y,t)$ function by $w(x,y)\sin(\omega t)$ and presented the in-plane force N_x with a linear function. More recently, Akhavan et al. [1] studied a similar case where closed-form solutions for the free vibration analysis of Mindlin plates with uniform and linearly distributed in-plane loading at two opposite edges with simply supported boundary conditions were

given for different configurations of boundary conditions at the opposite edges while the plate is resting on elastic foundation.

As seen in the previous section, closed-form solutions are mainly available in the literature for simple structural assemblies with basic boundary conditions and loading schemes. On the other hand, numerical approaches give acceptable approximations of the vibrational behavior of more complex systems. The Finite Element method approximates the flexural vibration of plates by dividing the plate into a mesh of two-dimensional finite elements (i.e. plate elements) in rectangular, triangular or quadrilateral shapes [26].

The element used for the FE modeling in this paper is the *generalized shell* [42] with four joints as shown in Fig. 1. The shell stiffness formulation is based on a four-point numerical integration where stiffness components are calculated for six degrees of freedom (DOFs) namely U1, U2, U3, R1, R2 and R3 at each node [28]. The internal forces of the element are evaluated at the 2-by-2 Gauss integration points and extrapolated to the element joints [42]. The term “generalized shell” is used for this element because it could be reduced into *plate* (i.e. pure bending) behavior with only the U3, R1 and R2 DOFs at each node, or *membrane* (i.e. no bending) behavior with stiffness components only at U1, U2 and R3 accordingly [42]. Regarding the transverse shearing deformations, there are two available options for the generalized shell element namely the *thick-shell* and the *thin-shell* where the element formulations include those deformations in the analysis in the former case based on Reissner-Mindlin plate theory [24,30] and neglecting them in the latter case based on Kirchhoff-Love plate theory [21]. Although shearing deformations tend not to affect the overall plate deflections considerably when the plate thickness is less than a decimal fraction of its dimensions, the developer of the generalized shell element [42] recommends using the thick-plate option for the FE analysis even for the case of thin-plate bending problems. The main issue involved with the modeling of relatively thin plates in FE analysis is the shear locking. There are several techniques in the literature to reduce or remove this phenomenon including the Reduced Integration method [15], the Discrete Shear Gap (DSG) method [5], the Isogeometric analysis [16], the hierarchic family of isogeometric shells [11] and the Modified Mindlin plate theory [33]. The generalized shell element in this paper is a locking-free element, which has been verified for the more complex case of orthotropic thin-walled beams by Minghini et al. [25].

The geometric nonlinearities (i.e. large displacement effects) are included in the analysis for the static step-by-step procedure in which the equilibrium equations are updated in the deformed configuration of the structure at the end of each step. Large displacements and rotations

Table 1
Various boundary conditions combinations for a panel of industrial duct walls.

Boundary Condition Combination	Edge Number			
	1	2	3	4
CC	Clamped	Clamped	Clamped	Clamped
CS	Clamped	Clamped	Simple	Simple
SS	Simple	Simple	Simple	Simple
SC	Simple	Simple	Clamped	Clamped

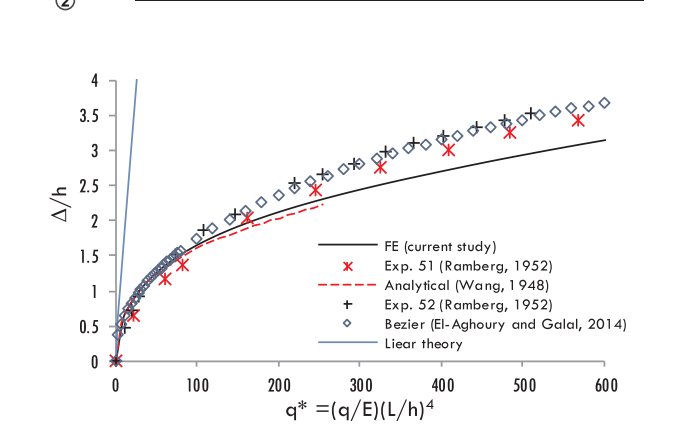


Fig. 2. First stage of the validation study: comparison of the results of FE analysis (current work) with available literature for the nonlinear load-deformation of rectangular plates under lateral uniform pressure.

are accounted for but strains are assumed to be small. The position of each element is tracked during the analysis using an updated Lagrangian formulation [42].

This paper considers four combinations of boundary conditions for the case of rectangular plates. These combinations are addressed as SS, SC, CS and CC as shown in Table 1.

There are three stages of verification of the FE analysis in this paper. In the first stage, as shown in Fig. 2, the load-displacement curves resulted from FE static analysis are compared to the results of two experimental tests by Ramberg et al. [29], one analytical solution by Wang [39] and one numerical solution by El Aghoury and Galal [12]. It is seen in the chart (Fig. 2) that the FE analysis is in a very good correlation with the experimental and analytical data in the literature. In the second stage, as shown in Fig. 3, the natural frequencies calculated by the FE modal analysis for the case of a rectangular plate are verified

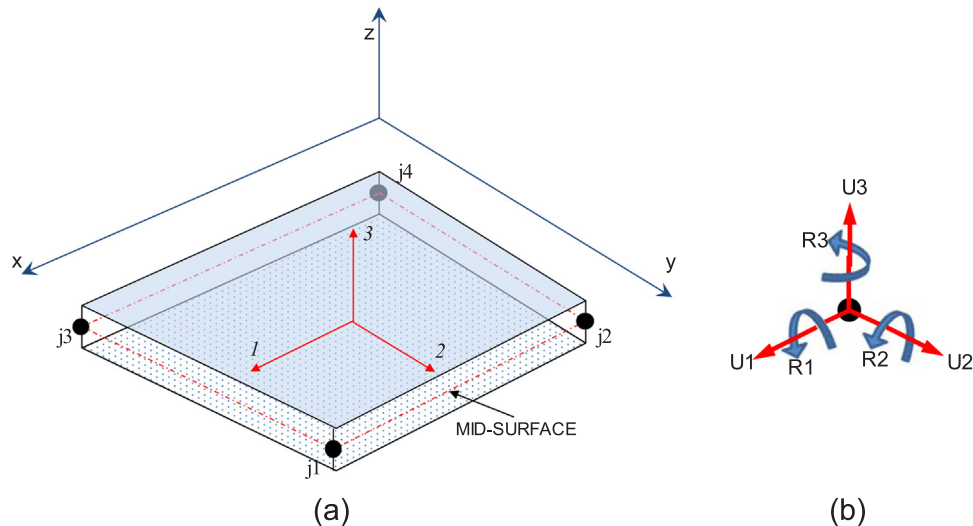


Fig. 1. (a) General view of the nodes, surfaces, global and local coordinates of the shell element (b) local degrees of freedom at a given joint.

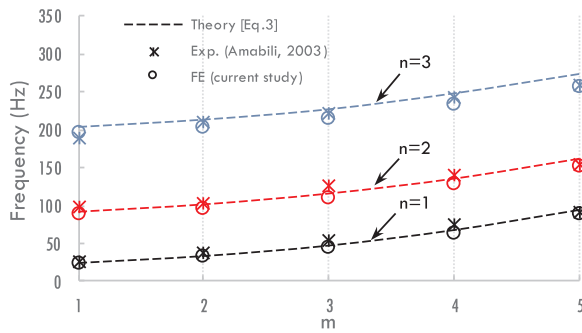


Fig. 3. Second stage of the validation study: comparison of the FE results (current work) with the natural frequencies of 15 modes of vibration of the plate tested by Amabili [2] and their reported analytical results.

against the experimental data and the closed-form solution to Eq. (3) by Amabili [2]. The results of FE analysis show an excellent correlation with the experimental and analytical results in the literature. In the third stage, as shown in Fig. 4, the vibration frequencies of two rectangular plates with different aspect ratios at four different levels of membrane forces are verified against the analytical results of Wang et al. [40]. Those four membrane force levels represent 0%, 50%, 80% and 95% of the plate critical buckling load. For the first plate with aspect ratio 1.0 (Fig. 4a), the vibration frequencies of seven modes are reported by Wang et al. [40] and for the second plate with aspect ratio 0.5 (Fig. 4b), there are six modes reported by Wang et al. [40]. The FE analysis was able to capture all of the reported modes with very good correlation. As seen in the charts, an excellent correlation between the results of FE analysis and the existing experimental, analytical and numerical results from the literature has been achieved for all the three stages of verification.

3. Free vibration of rectangular plates at rest

This section presents the modal analysis of rectangular plates at rest i.e. no loads are applied to the plate. To uncouple the analysis results from the plate dimensions, the *non-dimensional frequency parameter* (ω^*) which is used frequently in the literature is adopted here as shown in Eq. (5).

$$\omega^* = \omega(L/\pi)^2 \sqrt{\rho h/D} \quad (5)$$

As mentioned before, several works in the literature address the calculation of vibration frequencies of rectangular plates in various pre-defined mode shapes. Although the pre-defined mode shapes are limited by nature, some of the available closed-form solutions tend to calculate the vibration frequency for any given mode (see Eq. (3)). On the other hand, numerical techniques such as Finite Element method, allow for the calculation of vibration frequencies for unlimited

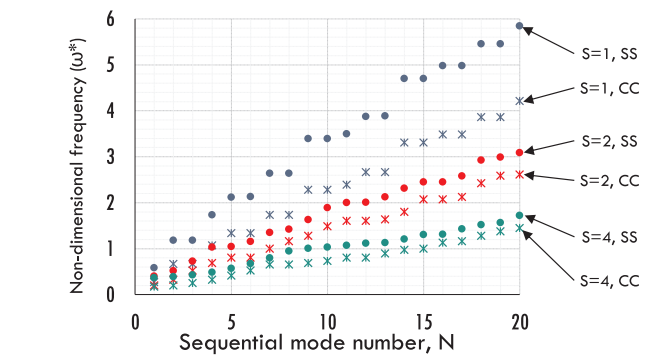
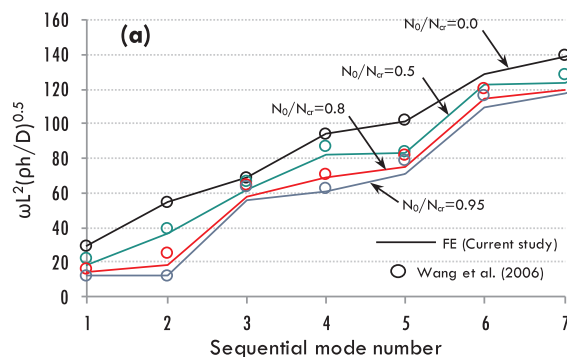


Fig. 5. Non-dimensional frequencies associated with the first 20 modes of vibration for simply-supported (SS) and clamped (CC) rectangular plates with aspect ratios $S = 1.0$, $S = 2.0$ and $S = 4.0$.

sequential modes by implementing verified methods such as Eigen and Ritz analysis [42]. As an example, the non-dimensional frequency parameter (ω^*) that is associated with the first 20 modes of free vibration of rectangular plates with 2 different boundary conditions combinations (CC and SS) and 3 different aspect ratios ($S = 1$, $S = 2$ and $S = 4$) are shown in Fig. 5.

3.1. Structurally significant modes

Each of the four nodes of the rectangular plate element in the FE models of this paper involves six DOFs (i.e. translation in and rotation about the global Cartesian axes X, Y and Z). Assuming the uniform distribution of the plate mass over these nodes, the nodal mass participation at each node for every individual vibration mode is determined as a result of the modal analysis. The overall mass participation of the plate at each DOF is then calculated as the summation of all individual nodal mass participations. As an example, the mass participation ratios of the first 20 modes of vibration of the rectangular plate with aspect ratio $S = 2.0$ and SS boundary conditions (see Fig. 5 for the frequencies) are shown in Table 2.

As seen in the table, the overall mass participation is zero for the three in-plane DOFs i.e. U(X), U(Y) and R(Z). This phenomenon is easily understandable considering the plate geometry and boundary conditions. As for the main DOFs i.e. U(Z), R(X) and R(Y), it is seen in Table 2 that the cumulative mass participation ratio reaches 0.884, 0.761 and 0.784 respectively with the consideration of 20 modes. In order to better visualize these mode shapes and their mass participation, they are displayed schematically in Fig. 6. It is obvious that higher cumulative mass participations are achievable by considering greater number of modes but it should also be noted that individual modes with great mass participations are already achieved in the three *significant* DOFs. In this case, they are mode 1 for U(Z), mode 2 for R(Y) and mode 4 for R

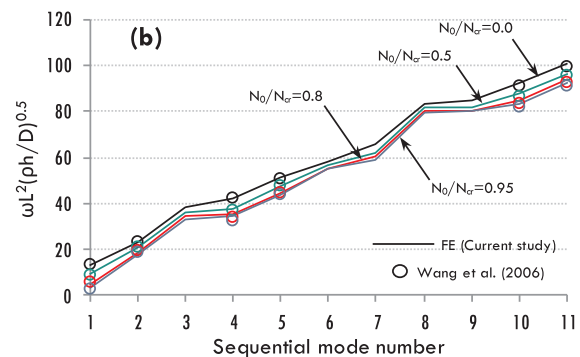


Fig. 4. Third stage of the validation study: non-dimensional frequency $[\omega L^2(\rho h/D)^{0.5}]$ of rectangular SC plates with uniaxial membrane force resulted from FE analysis (current work) against the analytical results of Wang et al. [40] (a) plate aspect ratio: 1.0 (b) plate aspect ratio: 0.5.

Table 2
Modal mass participation ratios at the six DOFs for the rectangular plate with aspect ratio $S = 2.0$ and SS boundary conditions.

Mode	ω^*	Modal mass participation ratio					
		UX	UY	UZ	RX	RY	RZ
1	0.20	0	0	0.682	0	0	0
2	0.32	0	0	0	0	0.524	0
3	0.52	0	0	0.076	0	0	0
4	0.68	0	0	0	0.537	0	0
5	0.80	0	0	0	0	0	0
6	0.80	0	0	0	0	0.131	0
7	1.00	0	0	0	0.059	0	0
8	1.16	0	0	0.027	0	0	0
9	1.27	0	0	0	0	0	0
10	1.48	0	0	0.075	0	0	0
11	1.59	0	0	0	0	0.115	0
12	1.59	0	0	0	0	0.004	0
13	1.63	0	0	0	0.021	0	0
14	1.79	0	0	0.008	0	0	0
15	2.07	0	0	0	0	0.014	0
16	2.07	0	0	0	0	0	0
17	2.11	0	0	0.014	0	0	0
18	2.42	0	0	0.003	0	0	0
19	2.58	0	0	0	0.011	0	0
20	2.59	0	0	0	0	0	0
Total:		0	0	0.884	0.761	0.784	0

(X) with mass participation ratios of 0.682, 0.524 and 0.537, respectively. Similar outcome for the mass participation ratios are obtained for the other boundary conditions combinations and aspect ratios. In all cases, the first two sequential modes correspond to the highest mass participation for U(Z) and R(Y) respectively. On the other hand, the third significant mode i.e. the mode with the highest mass participation along R(X) does not necessarily correspond to the third sequential mode. These three mode shapes are considered as the “significant modes” and will be the topic of the upcoming parametric studies in this paper.

3.2. Vibration frequencies of significant modes for rectangular plates at rest

The frequency parameters of rectangular plates with various aspect ratios between 1.0 and 4.0 are shown in Fig. 7 and Table 3 for the three

significant modes of vibration. Based on these charts, it is observed that although an increase in the plate aspect ratio tends to lower the fundamental frequencies, this effect diminishes as the aspect ratio reaches higher values. It is also seen in the charts of Fig. 7 that the plate aspect ratio has a greater effect on the 2nd significant frequency than those of the other two modes.

It should be noted that the values presented in Table 3 as the non-dimensional frequency parameters could be used for interpreting a reasonable approximation of ω^* for any plate aspect ratios by using a linear interpolation.

4. Free vibration of rectangular plates with uniform lateral pressure and large deformation effects

As discussed earlier in Section 1, the lateral load and in-plane forces affect the natural frequency of rectangular plates (See Eq. (1)). Although the effect of directly-applied in-plane forces on the vibration of plates is not studied in this paper, a correlation between the lateral pressure and in-plane forces will be the main parameter in this section.

The classical plate theory establishes the lateral load-displacement behavior of thin rectangular plates mainly based on the assumption of pure flexural behavior that follows a linear trend. This assumption along with other less significant assumptions involved with Kirchhoff's plate theory are used widely by the design engineers as the accepted design basis for thin-walled structures. On the other hand, regardless of the boundary conditions, if the maximum deflection of the plate exceeds a certain limit, the linear load-displacement assumption does not reflect the true behavior (see Fig. 2). This deflection limit is defined as a portion of the plate thickness e.g. $\Delta > 0.5t$ [18], $\Delta > 1.0t$ [36,37] and $\Delta > 0.2t$ [3] where Δ is the maximum deflection at the center of plate as shown in Fig. 8.

The nonlinear load-displacement behavior of thin plates with large deformations is governed by von-Kármán equations [38]. One of the main assumptions in von-Kármán equations is that all strain components are small and linear elasticity is applicable to the plate material. Kirchhoff's plate hypothesis is also applicable to von-Kármán equations indicating that the stresses normal to the middle surface of the plate are negligible and the strains are linearly distributed through the thickness of the plate. The main advantage of von-Kármán equations over the classical plate theory is the consideration of membrane stresses that generate in the plate as a result of large deformations. There are a

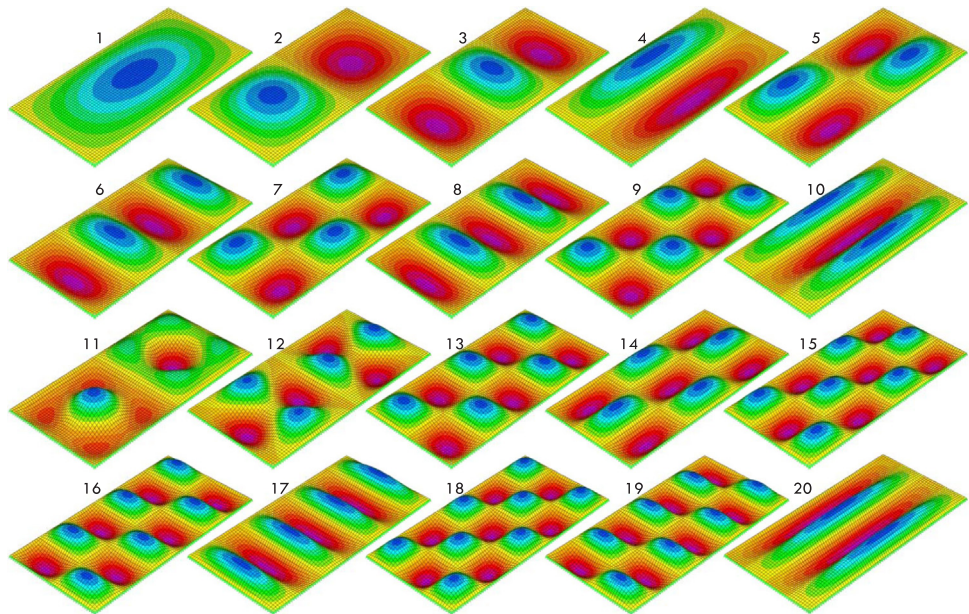


Fig. 6. Visual 3D display of the first 20 modes of vibration for a rectangular plate with aspect ratio $S = 2.0$ and SS boundary conditions.

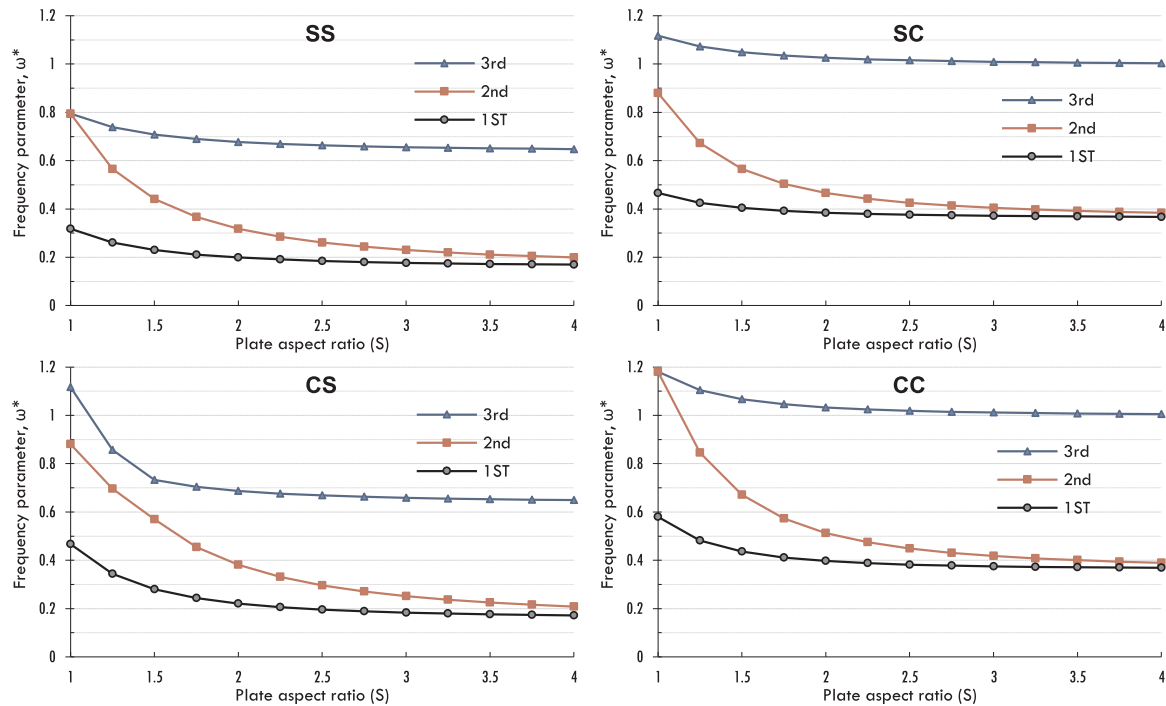


Fig. 7. Frequency parameter associated with the three significant modes for rectangular plates with various boundary conditions and aspect ratios.

number of experimental, analytical and numerical works on rectangular plates with geometrical nonlinearities based on von-Kármán equations including Ramberg et al. [29], Wang [39], Little [20], Young and Budynas [44], El Aghoury and Galal [12] and Rezaiefar and Galal [32] where the load-displacement and load-stress curves for various geometries and boundary conditions were presented.

A sequential analysis method is adopted in this section in order to study the effects of nonlinear load-deformation behavior on the free vibration properties of rectangular plates. The analysis consists of two stages for a series of uniform lateral pressure values. At the first stage, the uniform lateral pressure is applied to the plate and a nonlinear static large deformation analysis is performed. At the end of this stage, a modal analysis is performed on the plate and the frequencies of the three significant modes are recorded. This procedure is repeated for various lateral pressure values. The results of this analysis are presented in the charts of Fig. 9 as the non-dimensional frequency parameter against the unitless pressure $(P/E)(L/h)^4$ for various aspect ratios and boundary conditions combinations.

It is seen in the charts of Fig. 9 that the vibration frequencies of the three significant modes increase with the applied uniform pressure for

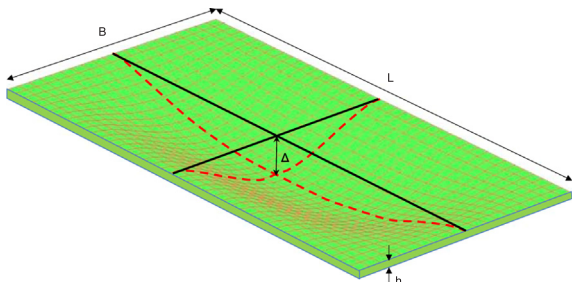


Fig. 8. Deformation of a CC rectangular plate under uniform lateral pressure.

all the studied cases. Although the charts presented in Fig. 9 could be used directly for specific cases, these charts are converted into more meaningful data by dividing the frequency parameter at each pressure by the value of the frequency parameter at rest (values presented in Table 3) and presented in the charts shown in Fig. 10.

Table 3
Values of ω^* associated with the three significant modes for rectangular plates with various boundary conditions and aspect ratios.

Aspect ratio (S)	SS			SC			CS			CC		
	1st	2nd	3rd	1st	2nd	3rd	1st	2nd	3rd	1st	2nd	3rd
1	0.3181	0.7946	0.7946	0.4668	0.8805	1.1172	0.4667	0.8805	1.1171	0.5797	1.1814	1.1814
1.25	0.2610	0.5659	0.7383	0.4253	0.6724	1.0729	0.3441	0.6958	0.8564	0.4819	0.8452	1.1044
1.5	0.2300	0.4417	0.7077	0.4043	0.5654	1.0492	0.2803	0.5696	0.7326	0.4357	0.6715	1.0669
1.75	0.2113	0.3668	0.6893	0.3922	0.5044	1.0351	0.2436	0.4549	0.7047	0.4112	0.5729	1.0459
2	0.1992	0.3182	0.6773	0.3847	0.4668	1.0260	0.2209	0.3813	0.6875	0.3969	0.5129	1.0331
2.25	0.1909	0.2849	0.6691	0.3796	0.4422	1.0198	0.2061	0.3314	0.6762	0.3879	0.4744	1.0247
2.5	0.1849	0.2611	0.6632	0.3761	0.4253	1.0154	0.1960	0.2962	0.6683	0.3820	0.4486	1.0189
2.75	0.1805	0.2435	0.6589	0.3735	0.4133	1.0121	0.1888	0.2706	0.6627	0.3778	0.4305	1.0147
3	0.1772	0.2301	0.6556	0.3716	0.4043	1.0096	0.1835	0.2514	0.6585	0.3748	0.4174	1.0116
3.25	0.1746	0.2196	0.6530	0.3701	0.3975	1.0077	0.1795	0.2367	0.6553	0.3726	0.4077	1.0093
3.5	0.1725	0.2114	0.6510	0.3689	0.3923	1.0062	0.1764	0.2252	0.6528	0.3709	0.4003	1.0074
3.75	0.1708	0.2047	0.6493	0.3679	0.3881	1.0049	0.1740	0.2160	0.6508	0.3695	0.3945	1.0059
4	0.1695	0.1992	0.6480	0.3672	0.3847	1.0039	0.1721	0.2086	0.6492	0.3685	0.3899	1.0047

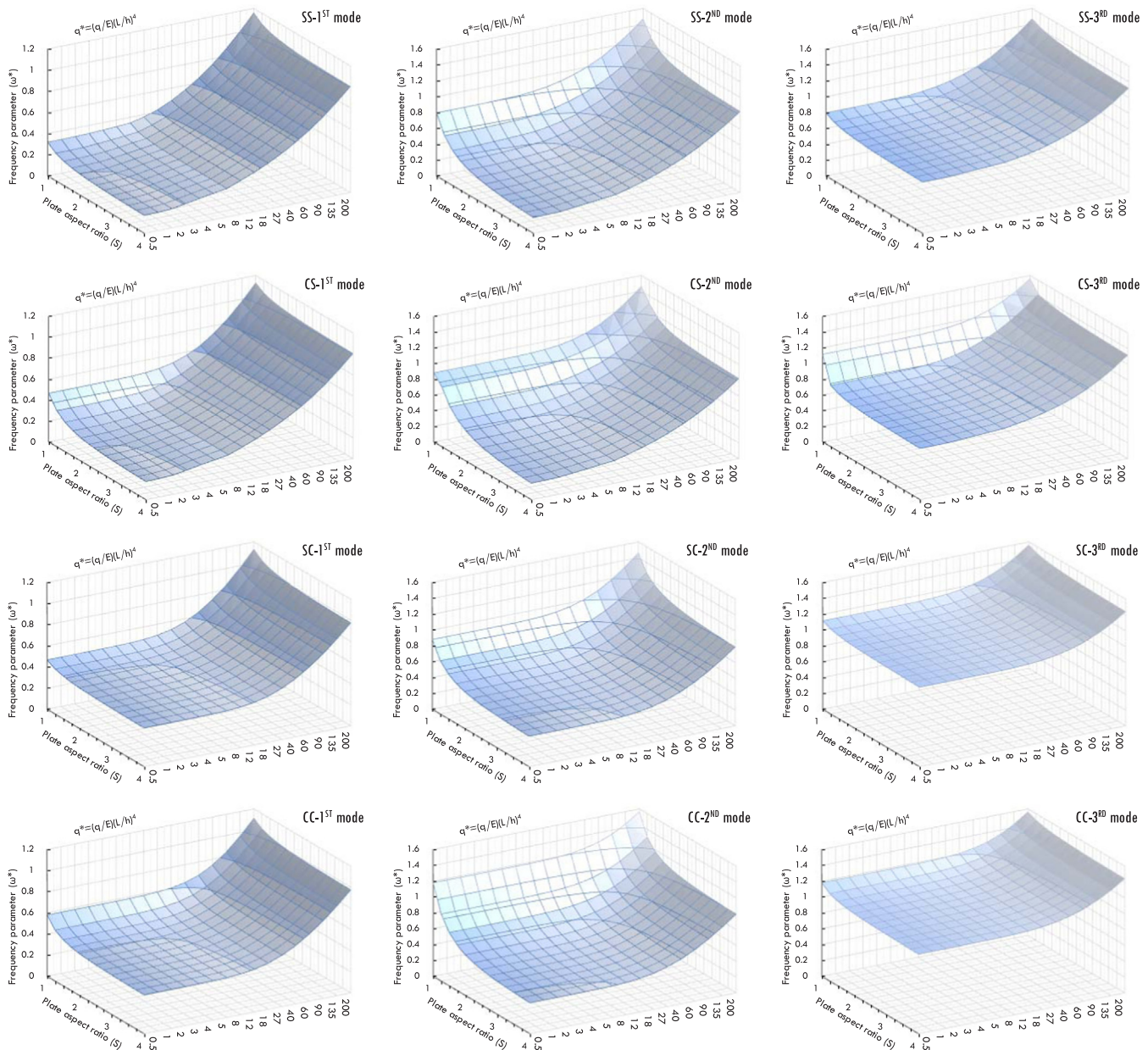


Fig. 9. Non-dimensional frequency parameter charts for plates with different boundary conditions and aspect ratios under various non-dimensional pressure (q^*) values.

5. Analysis example

The walls of an industrial duct segment consist of 8 mm steel plates ($F_y = 300$ MPa, $E = 200,000$ MPa, $\nu = 0.3$ and $\rho = 7850$ kg/m³) with relatively rigid stiffener ribs placed at 1.2 m intervals per the static analysis and design. The duct cross-section is a 3.0×3.0 square with fully welded corners. The internal negative pressure acting on the duct walls is 31.6 kPa. The working and cut-off frequencies of the fan are 90 Hz and 35 Hz, respectively. It is required to verify if the designed duct wall plates will cause resonance with the fan. A resonance margin of $\pm 20\%$ is to be considered for the study.

Solution:

Based on the problem definitions, each individual wall panel is a $3000 \times 1200 \times 8$ mm rectangular plate. The stiffened plate with fully welded duct corners suggests the CCSS boundary conditions.

Plate aspect ratio: $S = 3000/1200 = 2.5$

Plate flexural rigidity: $D = Eh^3/12(1 - \nu^2) = 9.377$ kN·m
 $(L/\pi)^2 \times (\rho h/D)^{0.5} = 0.0121$

The cut-off frequency of the fan corresponds to zero pressure on the walls thus, it is compared to the significant natural frequencies at rest based on the values from Table 3 as following:

$\omega^*(1st \text{ mode}) = 0.196$ Thus: $\omega_1 = 0.196/0.0121 = 16.2$ Hz

$\omega^*(2nd \text{ mode}) = 0.2962$ Thus: $\omega_2 = 0.2962/0.0121 = 24.5$ Hz

$\omega^*(3rd \text{ mode}) = 0.6683$ Thus: $\omega_3 = 0.6683/0.0121 = 55.3$ Hz

The problem requires the verification of fan frequencies with $\pm 20\%$ margins:

Working frequency: Lower limit: $90 - 20\% = 72.0$ Hz

Upper limit: $90 + 20\% = 108.0$ Hz

Cut – off frequency: Lower limit: $35 - 20\% = 28.0$ Hz

Upper limit: $35 + 20\% = 42.0$ Hz

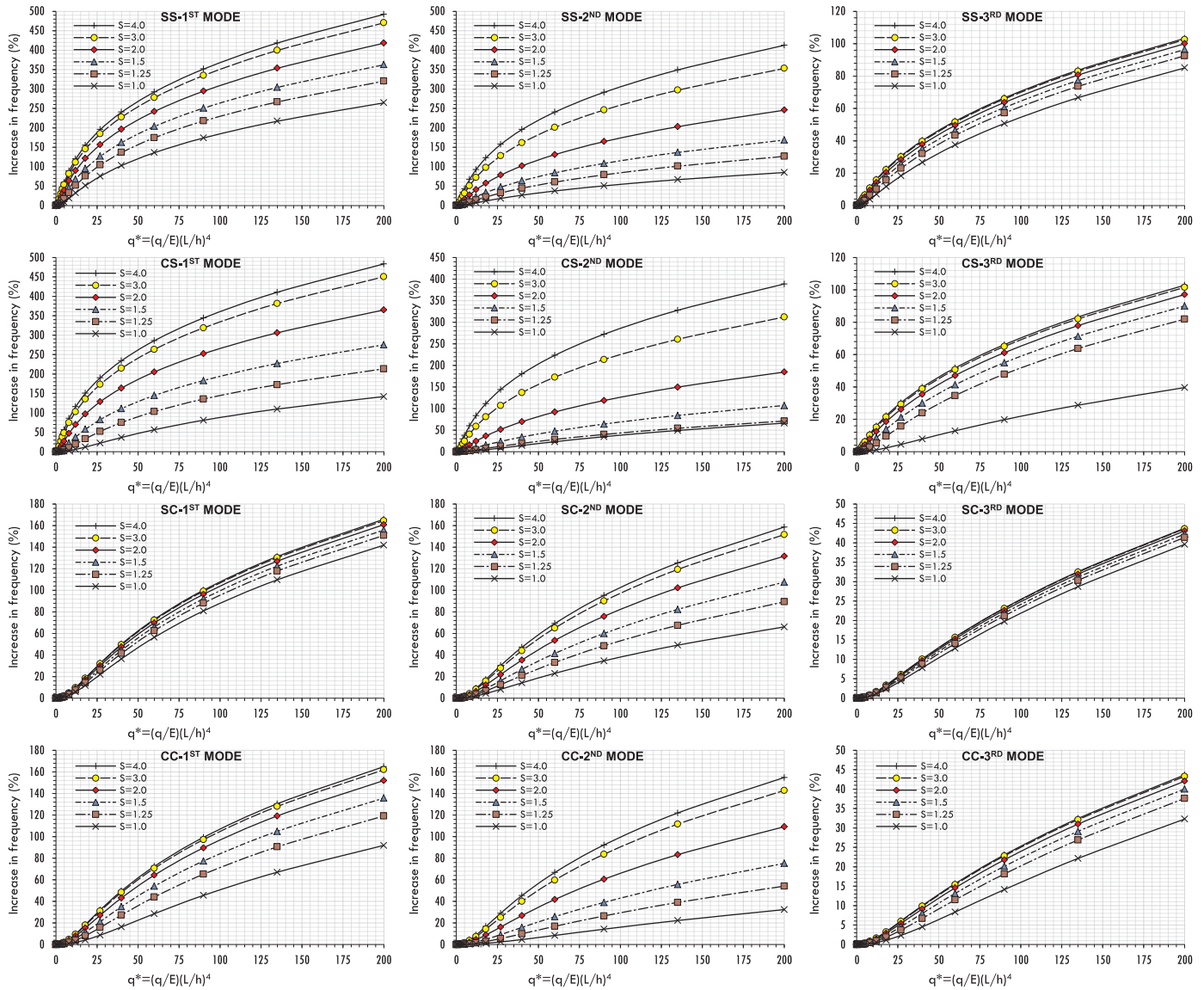


Fig. 10. Non-dimensional frequency parameter charts for plates with different boundary conditions and aspect ratios under various uniform lateral loads.

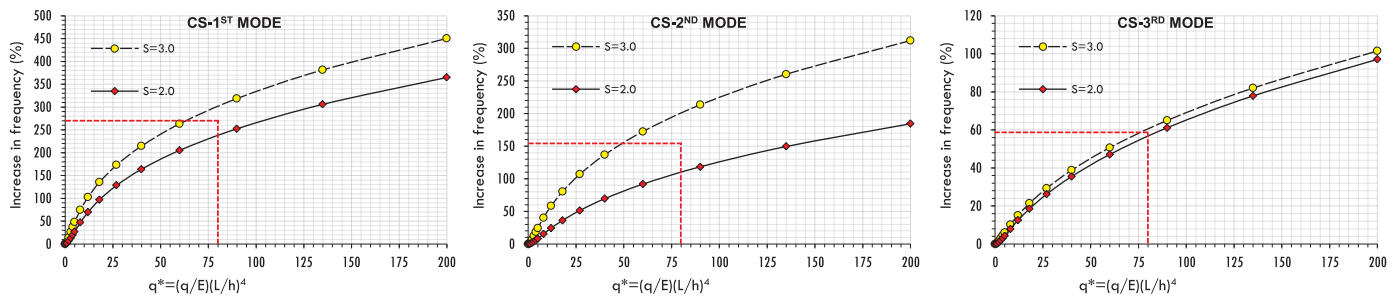


Fig. 11. Application of the pressure-frequency charts in an example.

It is seen that none of the three significant natural frequencies of the plate at rest falls within the 20% margins of the cut-off frequency of the fan.

At working conditions, the fan produces a uniform pressure of 31.6 kPa (0.0316 MPa) on the wall, thus:

Unitless pressure: $(P/E)(L/h)^4 = (0.0316/2e5)(1200/8)^4 = 79.99 \approx 80.0$

Using the charts from Fig. 10 to obtain the increase in frequencies as shown in Fig. 11, the frequencies at the 1st, 2nd and 3rd significant

modes should be increased by 270%, 155% and 59% respectively to account for the working condition of the duct walls:

$$\omega_{1(\text{work})} = 16.2 + 270\% = 59.9 \text{ Hz}$$

$$\omega_{2(\text{work})} = 24.5 + 155\% = 62.5 \text{ Hz}$$

$$\omega_{3(\text{work})} = 55.3 + 59\% = 87.9 \text{ Hz}$$

It is seen that the 3rd significant natural frequency of the plate falls within the 20% margins of the working frequency of the fan (i.e. $72.0 < 87.9 < 108.0$). This indicates that the plate is prone to

Table 4

Frequencies resulted from the method in this paper and case-specific FE analysis.

	P = 0.0 (rest)			P = 31.6 kPa		
	1st	2nd	3rd	1st	2nd	3rd
Proposed method in this paper	16.2	24.5	55.3	59.9	62.5	87.9
FE analysis	16.37	24.76	55.83	61.88	64.33	88.97
Discrepancy (%) ^a	1.05	1.06	0.96	3.3	2.93	1.22

^a Discrepancy = $100 \times (\omega_{FE} - \omega) / \omega$.

resonance with the fan at working conditions.

In order to verify these results, the plate in the example is also modeled using FE analysis. The frequencies at rest and at the given pressure (31.6 kPa) are listed in Table 4 along with their difference in percentage. It is seen that the results obtained from the proposed method in this paper are in excellent correlation with the results obtained from the FE analysis.

6. Concluding remarks

This paper presented the case of free vibration of rectangular plates with and without lateral pressure. It was shown that although an unlimited number of mode shapes could be considered for the vibration analysis of rectangular plates, there are only three modes with structural significance that need to be considered in the structural design. These three modes are:

1. The mode shape with one curvature along the breadth and length of the plate.
2. The mode shape with one curvature along the breadth and two curvatures along the length of the plate.
3. The mode shape with two curvatures along the breadth and one curvature along the length of the plate.

Non-dimensional vibration frequencies for these three significant modes were provided for four boundary conditions combinations and varying aspect ratios from 1.0 to 4.0. The frequency of vibration for any rectangular plate could be easily calculated by using these non-dimensional frequencies.

The effects of lateral pressure on the vibration frequencies of rectangular plates were studied in the last section of this paper based on the concept of large displacement analysis. It was demonstrated that for rectangular plates with various aspect ratios and boundary conditions combinations, the vibration frequencies increase significantly (up to 450%) with the lateral pressure. The charts and tables given in this paper for the vibration frequencies are all non-dimensional. Thus, they could be used for specific cases of rectangular panel plate design regardless of plate material, thickness and aspect ratios.

This work could be continued into the nonlinear vibration domain by studying the vibration interactions and the influence of loading frequencies on the dynamic response of the plate.

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