GETTING THE MOST OUT OF
YOUR SCIENTIFIC CALCULATOR

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INTRODUCTION

The modern scientific calculator is truly a marvel. Only those of us who grew up before the appearance of the calculator and have followed its evolution can fully appreciate what it can do. This booklet is written to tell you the things that textbooks, instructors, and even the instruction manuals of the calculators do not tell you, with special reference to the Sharp EL-531W and Casio *fx*-300MS calculators, which are good examples of modern (2009) scientific calculators available for less than $20 (Canadian). You should use this booklet with your calculator on hand, trying each of the examples to make sure you get the same result as we get. If you are not sure you understand, try some more examples. You can find suitable examples in your textbook (mathematics or other).

In what follows, each of the two calculators is treated separately. Whichever calculator you have, you should start (naturally enough) with the section "**Getting Started**", referring to the section "**Modes and Setup**" as necessary. Then your should work your way through the section "**Editing Expressions**", and continue with the sections "**Squares, Cubes, Reciprocals, Square Roots, Cube Roots**", "**Other Powers and Roots**", and "**Exponentials and Logarithms**". The examples relating to financial mathematics may or may not be familiar to you, but you should work through them to gain facility in using your calculator (and perhaps to learn some things about money!). You should at least go through the first part of the section "**Thanks for the Memories**". You will find the second part, "**M is for Memory**", especially interesting if you are taking Calculus II. The section "**Play it again, Sam**" is worth looking at, especially for users of the Casio, but not absolutely necessary. The section "**Playing the Angles**" is what you need if you are using trigonometric functions. The section "**Fractions, Proper and Improper**" will be helpful if you are doing row-reduction of matrices. The section "**Let Me Count the Ways…**" deals with factorials, permutations, and combinations. If you have the Sharp calculator and are involved with Computer Science or Computer Engineering, you will find the section "**Touching all the Bases**" useful. Finally, the section "**Vital Statistics**" will help you if you are taking a statistics course.

SHARP EL-531W

Getting Started

In the following, we will indicate keys to be pressed in boldface, thus:

2 + 2 =

and the results which you should see on the bottom line of the display in boldface italic, thus:

4.

You should try each of these examples, to make sure you get the same result.

Let's look at some simple arithmetic calculations. The calculator is turned on by pressing the **ON/C** key in the upper right-hand corner of the keyboard. Probably you will be able to follow through the rest of this part without any preliminary steps. But –

IF YOU FIND THAT YOUR CALCULATOR IS BEHAVING STRANGELY,
PROCEED TO THE SECTION "**Modes and Setup**" BELOW.

You should see

0.

at the right of the lower line of the display, and, in small letters at the very top of the display, "**DEG**" or "**RAD**" or "**GRAD**".

IF YOU SEE ANYTHING ELSE, PROCEED TO "**Modes and Setup**".

Now press the following keys:

3 + 4 × 5 =

You should see

3 + 4 × 5 =

in the first line of the display, and

23.

at the right of the second line. Notice that the calculators use normal conventions for evaluating mathematical expressions, with multiplications (and divisions) evaluated before additions and subtractions. If you are using some other type of calculator, and you see

35.

at the right of the second line, you are probably using a so-called "business calculator" (or perhaps a very old model). Get rid of it, and get yourself a scientific calculator.

Now press the following keys:

- 6 ÷ 7 =

You should see

ANS - 6 ÷ 7 =

on the first line, and

22.14285714

on the second line. What has happened is that the calculator assumes you want to take the previous result (namely, 23) as the starting point for the second calculation, and subtracted 6/7 from it. This will always happen when you start a new calculation with one of the keys +, -, ×, or ÷. It will also happen under certain other circumstances, which we will discuss below.

If you do not want this to happen, and you want to calculate -6/7 by itself, there are two methods. One is to clear out the previous calculation by pressing **ON/C ,** and then pressing

- 6 ÷ 7 =

giving

-0.857142857

as a result.

A better method is the following: Press

+/- 6 ÷ 7 =

giving the same result. (The key **+/-** is found just to the left of the **=** key in the bottom row.)
Notice the difference between the subtraction key **-** on the one hand, and the "change-sign" key
+/- on the other. The subtraction key **-** is used between two expressions to indicate that the
second is to be subtracted from the first. The "change-sign" key **+/-** precedes an expression to
indicate that its sign is to be changed (from negative to positive or from positive to negative).
Conceptually, these are two different operations, and you should get in the habit of using the one
you really want. Sometimes you can get away with using one in place of the other, but
sometimes you can't.

We saw that the calculator interprets 3 + 4×5 (correctly) as 3 + (4×5). But what if you
want to calculate (3 + 4)×5? One way is to press **3 + 4 =** to give **7.** and then **× 5 =** to give
35. as a result. As we have seen, your calculator will use the answer from the first calculation as
the starting point for the second. But what if you want $3 \times (4 + 5)$ or $\dfrac{3+4}{5+6}$? This is where the
parenthesis keys **(** and **)** come in handy. These keys are located above the keys **×** and **÷**, at the
right of the top row of grey keys. To calculate $(3 + 4) \times 5$, you press

(3 + 4) × 5 =

giving

$$35.$$

To calculate $3 \times (4 + 5)$, press

3 × (4 + 5) =

to give **27.** as the result. To calculate $\dfrac{3+4}{5+6}$, you have to realize that the fraction bar means that
the whole numerator, namely 3 + 4, is to be divided by the whole denominator, namely 5 + 6.
Thus, what you want is (3 + 4)/(5 + 6), so you punch it into the calculator accordingly:

(3 + 4) ÷ (5 + 6) =

giving **0.636363636** as the result. The same idea works for more complex calculations. To
calculate

$$\frac{\dfrac{3+4}{5+6}+\dfrac{9+8}{7+6}}{\dfrac{3+5}{4+6}-\dfrac{9+7}{8+6}},$$

you should think of the expression as

((3 + 4)/(5 + 6) + (9 + 8)/(7 + 6))/((3 + 5)/(4 + 6) - (9 + 7)/(8 + 6)),

and punch it in accordingly:

((3 + 4) ÷ (5 + 6) + (9 + 8) ÷ (7 + 6)) ÷ ((3 + 5) ÷ (4 + 6) - (9 + 7) ÷ (8 + 6)) =,

giving **-5.67016317** (try it!). In this case, it would be easier to perform the additions 3 + 4,
5 + 6, …, mentally, and then do the divisions, additions, and subtractions with the calculator, but
if the additions were, say, 3.4567 + 4.321, 5.987 + 6.123, …, you probably would want the
calculator to do the work. At any rate, the calculator can handle more levels of parentheses than
you are likely to toss at it.

It is not too hard to remember to put parentheses around the denominator when the denominator is a sum or a difference, as in the above example. But it is easy to forget parentheses when the denominator is a product or a quotient. Consider the following example:

$$\frac{3+4}{5\times6}.$$

It seems reasonable to punch this in as

(3 + 4) ÷ 5 × 6 =   NO!!!

giving **14.4** instead of 7/30 = 0.2333333333… . Why does this happen? To see the reason, consider the expression 7 - 5 + 6. You interpret this automatically as (7 - 5) + 6 = 8, not as 7 - (5 + 6) = -4. That is, in a sequence of mixed additions and subtractions, you perform the operations in sequence from left to right. You do not encounter sequences of mixed multiplications and divisions such as $7 \div 5 \times 6$ as frequently, but your calculator treats them in the same way as it treats mixed additions and subtractions. Thus, it interprets **(3 + 4) ÷ 5 × 6** as **((3 + 4) ÷ 5) × 6**, and arrives at the result **14.4**, as we have seen. But that is not what you want, so you have to punch in

(3 + 4) ÷ (5 × 6) =   YES!!!

to get the desired answer **0.233333333** .

One more thing. In ordinary algebra, when you put two expressions next to each other without any operation symbol between them, you understand that a multiplication is to be performed. Thus, when you write 3(4 + 5), you understand that this means $3 \times (4 + 5)$. The calculator will also understand this – sometimes. Thus, if you punch in

3 (4 + 5) =

you will get

27.

as you would expect. But if you punch in

(3 + 4) 5 =

you get

Error 1

as your result. Go figure! At any rate, to get rid of the error message, press **ON/C** . The best thing, once again, is to punch in what you really want, which is

(3 + 4) × 5 = .

Editing Expressions

But what if you make a mistake punching in your expression? One way out is to press **ON/C** and re-do the expression from the beginning. But if your expression is long, like the fraction example above, you may not want to do this. Fortunately, there is another way. Suppose you have punched in

3 + 4 × 5 =

and you realize that you really want to have

3 - 4 × 5 = .

This is where the arrow keys come in. They are located at the top of the keyboard, just below the display. We will denote them by $\Leftarrow$, $\Uparrow$, $\Rightarrow$, and $\Downarrow$. For the moment, we consider only the left arrow ($\Leftarrow$) and right arrow ($\Rightarrow$). (We discuss the up arrow ($\Uparrow$) and down arrow ($\Downarrow$) in the sections "**Play it again, Sam**" and "**Vital Statistics**" below.)

If you punch in the keys **3 + 4 × 5 =** , you will see on the top line

3 + 4 × 5

and at the right of the bottom line

23.

If you press ⇐ (the left arrow key), the entry in the bottom line will be replaced by **0.** , and the **=** at the end of the top line will be replaced by an underline (**_**). At this point, you can add more symbols to the expression. If, for example, you press **+ 6 × 7 =** , the top line becomes

3 + 4 × 5 + 6 × 7 =

and on the bottom line you will have the new result

65.

Now suppose you want to replace the first **+** by a **-** , so that your expression will be **3 - 4 × 5 + 6 × 7** . Press ⇒ (the right arrow key). This time, there is a flashing cursor over the **3** , either a left-pointing triangle (the most likely), or a rectangle. Press ⇒ a second time. Now the **+** will alternate with the flashing cursor. If your cursor is a triangle, press the **DEL** key. (This key is to the right of the arrow keys, in the third row, below the **MODE** key.) The **+** will disappear, and the rest of the expression will move left to fill its place. Now, press the **-** key. A minus sign (**-**) will be inserted between the **3** and the **4**. Finally, press the **=** key. The top line should now read

3 - 4 × 5 + 6 × 7 =

and the bottom line should read

25.

But what if you have the flashing rectangle cursor? In this case, when you press the **-** key, the minus sign will replace the **4** . If you press other keys, you will see that the keys you press will write over the numbers and operation signs you already have. Sometimes this is what you want, but sometimes it is not. To set your calculator so that the numbers and operation signs are inserted between the numbers and operation signs already in the expression, press **2nd F** (at the top left of the keyboard, to the left of the arrow keys), and then press the **DEL** key. (Note that **INS** is written, in orange, at the upper left of the **DEL** key.) After a short pause, the flashing rectangle cursor will be replaced by the flashing triangle, and the keys you press will be inserted into the expression between the existing numbers and operation signs.

You can flip back and forth between the triangular cursor (which inserts symbols into existing expressions) and the rectangular cursor (which writes over existing symbols) by pressing **2nd F DEL** (**INS**). The calculator will stay in whichever mode you choose, even if you turn the calculator off and on, until you press **INS** again.

Two things stay unchanged: First, whether you have the triangular cursor or the rectangular cursor, when you press the **DEL** key (without the **2nd F** key), you will always delete the symbol under the cursor. Second, to start making additions or changes to the *right* end of the expression, press ⇐ (the *left* arrow key). To start making changes to the *left* end of the expression, press ⇒ (the *right* arrow key). The idea seems to be that you press ⇐ if you want to make corrections from right to *left*, and ⇒ if you want to make corrections from left to *right*.

To make more changes, move the flashing cursor to the right or left, using the arrow keys, and make whatever changes you like. Note that the length of the expression is not limited by the twelve characters that can be displayed at any one time. The display will "slide along" so that the cursor is always in view, to a maximum of 142 characters. This should be enough for any reasonable calculation.

Experiment with making insertions and deletions until you feel comfortable with this aspect of your calculator's operation.

Squares, Cubes, Reciprocals, Square Roots, Cube Roots

Your calculator can calculate squares (x^2), cubes (x^3), reciprocals ($\frac{1}{x}$), square roots ($\sqrt{x}$), cube roots ($\sqrt[3]{x}$), as well as arbitrary powers (y^x) and roots ($\sqrt[x]{y}$). You will find keys marked √, **x^2**, and **x^3** as the second, third, and fourth keys in the second row of black keys. To get reciprocals, you press **2nd F** and then the **x^2** key. (Note that the **x^2** key has **x^{-1}** above it.) To get cube roots, press **2nd F** and then the **x^3** key (which has **$\sqrt[3]{x}$** above it).

The operation of these keys is straightforward. To get squares, cubes, and reciprocals, you press the corresponding key *after* entering the number to be squared, cubed, or inverted. Thus you have, for squares,

5 x^2 =

25.

for cubes,

5 x^3 =

125.

and for reciprocals,

5 2nd F x^2 =

0.2

To get square roots and cube roots, you press the corresponding key *before* entering the number. Thus we have, for square roots,

√ 5 =

2.235067077

and for cube roots,

2nd F x^3 =

1.709975947

Of course, it is never necessary to use the reciprocal function key to find the reciprocal of a number. You can always find $1/x$ in the obvious way, by pressing **1 ÷**. However, it is sometimes convenient to use the reciprocal function. For example, to calculate $1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7 + 1/8$, you can press

1 + 2 $^{-1}$ + 3 $^{-1}$ + 4 $^{-1}$ + 5 $^{-1}$ + 6 $^{-1}$ + 7 $^{-1}$ + 8 $^{-1}$ =

to get **2.717857143** .

If you do not specify the number whose square, cube, reciprocal, square root, or cube root you want, the calculator will assume that you want to use the answer from the previous calculation, if there is one. Thus, if you have calculated **3 + 4 × 5 =** to get a result of **23.** , and you press **x^2** , you will see **ANS^2** in the first line of the display. If you then press **=** , you will see **529.** as your answer; similarly if you press **x^3** or **x^{-1}** . Likewise, again supposing that your previous answer was **23.** , if you press **√ =** , you will see **√ANS** on the first line, and **4.795831523** on the second line; similarly if you press **^{3}√ =** .

Example: Solving Quadratic Equations

As you know, the roots of a quadratic equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \ .$$

You can calculate the roots using this formula, remembering that the numerator, the denominator, and also the discriminant b^2 - 4*ac* must be enclosed in parentheses for calculation.

    Problem: Find the roots of the equation $3x^2 + 4x - 5 = 0$.

    Solution: You can calculate one of the roots as follows:

    (± 4 + √ (4 x^2 - 4 × 3 × ± 5)) ÷ (2 × 3) =

giving **0.786299647** . You can easily calculate the other root by pressing ⇒ , then pressing ⇒ repeatedly until the cursor is on the **+** sign, pressing **-** to replace **+** by **-** , then pressing **=** to get **-2.119632981** .

    If the equation has no real roots, because the discriminant b^2 - 4*ac* is negative, you will get an error message **Error 2** . In this case, you can press the left or right arrow keys to bring you to the point in the expression where the error lies (or at least where the calculator thinks the error lies – sometimes the cause of the error lies elsewhere).

Other Powers and Roots

    To calculate other powers, you use the **y^x** key. For example, to calculate 3^5, you press

    3 y^x 5 =

to get **243** . Both the base and the exponent can be decimal fractions. For example, if you press

    3 . 4 5 y^x 6 . 7 8 = ,

you get **4430.099337** .

    The base can be a negative number, provided that the exponent is an integer (whole number, positive or negative):

    (± 3 . 4 5) y^x 6 =

gives **1686.221298** . Note the parentheses! If you press

    ± 3 . 4 5 y^x 6 =

the calculator will insert parentheses around the -3.45 and will interpret the expression as $(-3.45)^6$, giving **1686.221298**, which may or may not be what you want. If you want -(3.45^6), press

    ± (3 . 4 5 y^x 6) =

giving **-1686.221298**. Once again, the remedy is to punch in what you want, and not leave it up to the calculator to decide what it thinks you want.

    The base can also be negative when the exponent is a fraction whose denominator is odd. For example, you can have

    ± 5 y^x (1 ÷ 3) =

giving **-1.709975947** . You can even replace **(1 ÷ 3)** by **3 $^{-1}$** . Try it! But

    (± 5) y^x . 3 3 3 =

gives **Error 2**, because the calculator interprets .333 as 333/1000. Likewise, the exponent can be negative:

    3 . 4 5 y^x ± 6 . 7 8 =

gives **0.000225728** . Note that the calculator puts parentheses around the **-6.78** in the display. If you like, you can always calculate squares, cubes, reciprocals, square roots, and cube roots using the exponents 2 , 3 , -1 , 1/2 , and 1/3 . But it is quicker and easier to use the special keys provided for these frequently-occurring powers.

The procedure for calculating roots is much the same. To calculate $\sqrt[5]{3}$ (the fifth root of 3), you press

5 **2nd F** y^x **3** **=**

to get **1.24573094** . This works exactly the same way as $3^{(1/5)}$, and the same remarks concerning negative numbers hold.

You can also use the answer from the previous calculation in these functions. Suppose again that **23.** is the result of the previous calculation. If you press y^x , you will see **ANS^** in the first line of the display. If you now press (for example) **4 =** , you get **279841.** as your answer, since $23^4 = 279841$. But what if you want to use the previous answer as our exponent, say to calculate 2^{23}? You start out by pressing

2 y^x

in the usual fashion. You can then call up the previous answer to serve as the exponent. Press **ALPHA** (the blue-green key at the top left, just below the **2nd F** key), and then the **=** key. (Note that **ANS** appears in blue above and to the right of the **=** key.) But watch out! If by mistake you press the **2nd F** key instead of the **ALPHA** key, you will see a small raised ^{P} at the right of the display, and your calculator will start behaving very strangely. To get out of this predicament, press **2nd F** **+** , putting your calculator back into decimal mode, which is what you want. (Note that **→DEC** appears above the **+** key.) You can find out more about this in the section "**Around the Bases**" below.

Similarly, you can use the previous answer in a root calculation. Suppose again that **23.** is the result of the previous calculation. It is unlikely that you would want to calculate the 23rd root of some expression, but if you did, you could just press **2nd F** y^x to put **ANS** in the expression. More likely, you would want to calculate some root of the previous answer. Say you wanted the fifth root of your previous answer. You would then press

5 **2nd F** y^x **ALPHA** **=** **=**

to get **1.872171231** as the new answer.

Example: Compound Interest

The formula for compound interest is $A = P\left(1+\dfrac{r}{n}\right)^{nt}$, where A is the final Amount, P is the starting Principal, r is the annual Rate of interest, n is the Number of times interest is compounded per year, and t is the Time in years.

Problem: What will be the value at maturity of an investment of $10000 at 7% interest for 5 years, if interest is compounded (a) yearly (b) quarterly (c) monthly (d) daily (e) hourly?

Solution: Here you have $P = \$10000$, $r = 0.07$, and $t = 5$. In parts (a), (b), (c), (d) you have $n = 1$, $n = 4$, $n = 12$, $n = 365$, and $n = 24 \times 365$ respectively. You punch the numbers into the calculator just as you see them. In part (a), you don't need to punch in the 1 for n, but you might as well do it anyway, because it will be useful for the later parts:

10000 **×** **(** **1** **+** **0.07** **÷** **1** **)** y^x **(** **1** **×** **5** **)** **=**

giving **14025.51731**. Thus, the value at maturity is $14025.52 (rounding off to the nearest cent). For part (b), all you have to do is to edit the expression, replacing $n = 1$ by $n = 4$, giving

10000 **×** **(** **1** **+** **0.07** **÷** **4** **)** y^x **(** **4** **×** **5** **)** **=**

giving **14147.78196**, or $14147.78. Replacing 4 by 12, you get **14176.2526**, or $14176.25; replacing 12 by 365, you get **14190.19929**, or $14190.20. For part (e), you replace 365 by (24×365) (don't forget the parentheses!), so that you have

10000 × (1 + 0.07 ÷ (24 × 365)) y^x ((24 × 365) × 5) =

giving **14190.65559**, or $14190.66. Notice that you only need to change the values of n, and you do not need to calculate 24×365 separately. You can, and should, leave that to the calculator. If you feel like it, you can try calculating the amount if interest is compounded "minutely", that is, every minute.

In general, you should try to do your calculations using only a single expression, without writing down any intermediate results. The following example (taken from real life) shows what can happen when you don't take full advantage of what the calculator can do. Suppose you have the previous example, with monthly compounding. You punch in

1 + 0.07 ÷ 12 =

and get **1.005833333**. You then write down (incorrectly) 1.0583 on your paper, and punch in

1.0583 y^x 60 =

to get **29.95903136**, and finally

× 10000 =

to get **299590.3136**, or $299590.31. Your $10000 has grown to almost $300000 in just 5 years! How did you do it? Your first mistake was to miscopy your result from the first calculation. But that is not all. You might have copied it down correctly onto your paper, but punched it back into the calculator wrong. If you had avoided these mistakes, but still used the rounded-off value 1.00583, punching in

1.00583 y^x 60 =

you would have gotten **1.417343406**; multiplying by 10000 would give **14173.43406**, or $14173.43. This is pretty close to the right answer, but it is still not right. If you had not rounded off you answer, and used 1.005833333, you would have gotten the same answer as above, but at the cost of additional effort, and running the risk of making a slip, either in writing the number down or in punching it back into the calculator. The point is that you paid for the calculator (at least I hope you did!), and you might as well take full advantage of its capacities. The simplest and safest thing is to do the calculation in a single step.

You may ask "If I don't write any intermediate values on my paper during an exam, how does the marker know how I got the answer?" My answer is that you can write the key-strokes on your paper, just as you have been doing in these examples. If that doesn't satisfy the marker, nothing will.

Notice that as n increases, the final amount A increases, but it increases more and more slowly, so that it seems to be approaching a fixed value. We will discuss this in more detail when we discuss "continuous compounding" in the section "***Exponentials and Logarithms***" below.

Here is an example showing how to solve for P, when you are given A:

Problem: How much do you need to invest now in order to have $5000 three years from now, with interest at 4.5% compounded monthly?

Solution: You are given $A = \$5000$, $r = 0.045$, $n = 12$, and $t = 3$; you want to solve for P.

There are two approaches. One is to calculate $\left(1+\dfrac{r}{n}\right)^{nt}$, and then divide the result into A. Doing it this way, you punch in

(1 + 0.045 ÷ 12) y^x (12 × 3) =

giving **1.144247832** . Then you punch in

5000 ÷ ALPHA = =   (that is,   **5000 ÷ ANS =**   )

giving **4369.682738**, or $4369.68.

The other way is to do the whole thing in one calculation, punching in

5000 ÷ (1 + 0.045 ÷ 12) y^x (12 × 3) =

giving the same answer. The advantage to the second method is that if you want to re-do the calculation using, for example, $r = 5\%$, you only need to change 0.045 to 0.05 without having to punch in the rest of the expression.

In the section "**Exponentials and Logarithms**" below you will see how to solve for r and for t when the other variables are given.

Example: Future Value of a Series of Payments

The formula for the future value of a series of payments is

$$FV = PMT \frac{\left(1 + \dfrac{r}{n}\right)^{nt} - 1}{\dfrac{r}{n}},$$

where FV is the Future Value of the series of payments (with accumulated interest), PMT is the amount of each periodic PayMenT, and r, n, t are as before, namely, the annual interest Rate, the Number of times interest is compounded per year, and the Time in years.

Problem: Suppose you deposit $100 per month into an account paying 4.5% compounded monthly. How much will you have in the account after 5 years?

Solution: Here you have $PMT = \$100$, $r = 0.045$, $n = 12$, and $t = 5$. You punch the numbers into the calculator just as you see them in the formula, making sure to insert parentheses properly:

100 × ((1 + 0.045 ÷ 12) y^x (5 × 12) - 1) ÷ (0.045 ÷ 12) =

giving **6714.555214** , for an answer of $6714.56. It is always a good idea to check to see whether your answer is reasonable. In this case, you deposited $100 per month for 60 months, so you should have something over $6000. The extra $714.56 is the interest you have received during the five years. This looks reasonable. If the calculator had given $671.46 or $67145.55, for example, you would conclude that the answer is unreasonable, and you would check the calculation to see what had been punched in wrong.

Example: Monthly payment on a Loan

The formula for the present value of a series of payments to be made in the future is

$$PV = PMT \frac{1 - \left(1 + \dfrac{r}{n}\right)^{-nt}}{\dfrac{r}{n}},$$

where PMT, r, n, t are as above, and PV is the Present Value of the payments.

Problem: You borrow $5000 at 9% interest for three years to buy a car. What are your monthly payments?

Solution: Here you have $PV = \$5000$, $r = 0.09$, $n = 12$, and $t = 3$; you want to solve for *PMT*. As in the second compound interest example above, you have two choices. You can either calculate the messy algebraic expression on the right of the formula, and then divide it into

5000, using the ANS function, or you can calculate the whole thing at once, dividing 5000 by the expression on the right. Using the first method, you punch in

(1 - (1 + 0.09 ÷ 12) y^x (± 12 × 3)) ÷ (0.09 ÷ 12) =

giving **31.44680525** . You then press

5000 ÷ ALPHA = =

to get **158.9986633** , or $159.00 as your monthly payment. Using the second method, you punch in

5000 ÷ ((1 - (1 + 0.09 ÷ 12) y^x (± 12 × 3)) ÷ (0.09 ÷ 12)) =

giving the same result.

Let's check to see whether this is reasonable: For the 36 months, according to this, you pay a total of $159.00 × 36 = $5724.00. This covers your $5000 loan, with $724 interest. This seems about right. If you had calculated a monthly payment of $15.90 or $1590.00, you would see that there was something wrong.

Exponentials and Logarithms

Keys for Functions and Their Inverses

Your calculator has two keys for finding exponentials and logarithms. One of them, marked **log** , gives logarithms to the base 10; above it, in orange, you find **10^x** . The other, marked **ln** , gives logarithms to the base $e = 2.718…$; above it, in orange, you find **e^x** . Just as in the case of the function $\sqrt[3]{}$, the functions **10^x** and **e^x** are accessed by pressing **2nd F log**, and just as in the case of the functions **x^3** and $\sqrt[3]{}$, the functions on the same key "undo each other", or, in technical language, are *inverses* of each other. Thus, if you press, say, **3.45 =** and then **x^3 =** , you get **41.063635** . If you then press **2nd F x^3 =** , you get **3.45** back as a result. Likewise, if you press **3.45 =** and then **2nd F x^3 =** , you get **1.511029808** ; if you then press **x^3 =** , you get **3.45** back. Similarly, if you press **3.45 =** and then **log =** , you get **0.537819095** ; if you then press **2nd F log =**, you get **3.45** back. The same thing happens for the **ln** key. You also see the same relationship on the **sin** , **cos** , and **tan** keys, where the **2nd F** key gives the inverse trig functions. The only exception to this pattern is with the functions **x^2** and √ , which are inverses of each other, but occupy separate keys. The reason undoubtedly is that both **x^2** and √ are used so often that it saves time for the user not to have to press **2nd F** for one of them. (On some calculators, the key corresponding to **2nd F** is marked **INV** , for Inverse.) From now on, we will write $\sqrt[3]{}$, 10^x , e^x instead of writing out **2nd F** each time.

Entering Numbers in Scientific Notation

This is the place to mention the procedure for punching in numbers in scientific notation. One way to enter, for example, $6.02 × 10^{23}$ is to press

6 . 0 2 × 1 0 y^x 2 3 = .

A simpler way is to use the 10^x function:

6 . 0 2 × 10^x 2 3 = .

But the simplest and best way is to use the **Exp** key. This is located at the left of the third row of black keys. So press

6 . 0 2 Exp 2 3 =.

When you do this, you will see in the display

6.02ᴇ23=

$$6.02_{x10}{}^{23} \, .$$

If you want to enter an exact power of 10, say 10^{-33}, it is even simpler: Press

Exp ± 3 3 =.

You will see **1.$_{x10}{}^{-33}$** in the display.

Why *e*? Why Natural Logarithms?

In the old days (before 1980), we used tables of logarithms to the base 10. The way it worked was this: To find, say, 23.45×345.6, we wrote 23.45 and 345.6 in scientific notation as 2.345×10^1 and 3.456×10^2. Our tables of logarithms had logarithms of numbers between 1.000 and 9.999; the values of the logarithms were between 0.0000 and 0.9999. So we looked up the logarithms of 2.345 and 3.456, finding 0.3701 and 0.5386. The logarithms of 23.45 and 345.6, taking the powers of 10 into account, were 1.3701 and 2.5386. We then added these logarithms, getting 3.9087. We looked in our table to find the number whose logarithm was 0.9087, namely 8.104. Thus, our answer was 8.104×10^3, or 8104. (to four significant figures). (There were other complications, involving what were called "proportional parts", but I won't bore you with them.) The whole process was extremely unpleasant (ask your parents and grandparents!), and the results were only good to four significant figures. Fortunately, all this is obsolete. Logarithms are still important, partly because they are now so easy to calculate, but logarithms to the base 10 are rarely if ever used. Instead, we use logarithms to the base $e = 2.718\ldots$. In fact, if you see "log" in any advanced textbook, say beyond the first-year university level, you should assume that logarithms to the base *e* are meant, unless there is some statement to the contrary. These logarithms are called "natural logarithms", and often denoted by "ln", but it is not obvious at first glance what is "natural" about them.

It turns out the number *e* comes up in many different situations, any one of which could serve as its definition. Here is one which is convenient for us: If you calculate the value of the expression $(1 + x)^{1/x}$ for values of *x* close to 0, you can get a table like the following:

x	$(1 + x)^{1/x}$
-0.1	2.867971991
-0.01	2.731999026
-0.001	2.719642216
-0.0001	2.718417755
-0.00001	2.718295420
-0.000001	2.718283188
…	…
0.000001	2.718280469
0.00001	2.718168237
0.0001	2.718145927
0.001	2.716923932
0.01	2.704813829
0.1	2.593742460

You see that as *x* gets close to 0, the values of $(1 + x)^{1/x}$ seem to be getting close to a value which is approximately $2.71828\ldots$. It can be shown mathematically that this is the case, and that the number, to a better approximation, is $2.718281828459\ldots$. The German-Swiss mathematician Leonhard Euler (pronounced "Oiler") was the first to recognize the importance of this number

and give it a name. He chose to call it e, undoubtedly thinking that it would be immodest for him to give it the name E, as if he were naming it after himself.

A good example of the use of this number as a base of exponentials is given by compound interest. You recall that the formula for compound interest is $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$. Using the laws of exponentials, you can write this formula as $A = P\left[\left(1 + \dfrac{r}{n}\right)^{\frac{n}{r}}\right]^{rt}$. As n gets large, that is, as you compound more and more frequently, the value of $\dfrac{r}{n}$ gets close to 0, and the value of $\left(1 + \dfrac{r}{n}\right)^{\frac{n}{r}}$ gets close to e, since $\dfrac{n}{r} = \dfrac{1}{\dfrac{r}{n}}$. The formula $A = Pe^{rt}$ is gives what is called "continuously compounded interest", that is, the limiting value towards which the compound interest tends as you compound interest more and more frequently.

Example: Continuously Compounded Interest

Problem: What is the value at maturity of a loan of $10000 at 7% interest compounded continuously for 5 years?

Solution: We saw above that the final amounts for interest compounded yearly, quarterly, monthly, daily, and hourly were respectively $14025.52, $14147.78, $14176.25, $14190.20, and $14190.66 respectively. Using the formula for continuously compounding interest, we have $A = 10000 \times e^{(0.07 \times 5)}$. You can punch this into the calculator just as you see it:

10000 × e^x (0.07 × 5) = ,

giving 14190.67549 , or $14190.68. Thus, the value at maturity for continuously compounding interest is only $0.02 more than for hourly compounding, and only $0.48 more than for daily compounding.

You can also solve for r and t, using natural logarithms.

Problem: How long will it take for a principal of $5000 to increase to $8000 at 5.5% interest compounded continuously?

Solution: You have $A = \$8000$, $P = \$5000$, and $r = 0.055$. So the formula for continuously compounded interest gives $8000 = 5000 \times e^{0.055t}$, and you want to solve for t. Dividing by 5000, you get $\dfrac{8000}{5000} = e^{0.055t}$. Now, as we have said, ln x is the inverse function of e^x, so you can get rid of the exponential with base e by taking the natural logarithm of both sides: $\ln\left(\dfrac{8000}{5000}\right) = 0.055t$. Now, to solve for t, all that remains is to divide both sides by 0.055, giving $t = \dfrac{\ln\left(\dfrac{8000}{5000}\right)}{0.055}$. You can punch this into the calculator just as you see it:

ln (8000 ÷ 5000) ÷ 0.055 = ,

giving **8.545520532** , or about 8.55 years.

Example: Solving for *t*

Now let's see what happens when you use interest compounded monthly instead of continuously:

Problem: How long will it take for a principal of $5000 to increase to $8000 at 5.5% interest compounded monthly?

Solution: This time, you have, besides the above, $n = 12$. So the ordinary compound interest formula gives $8000 = 5000\left(1 + \dfrac{0.055}{12}\right)^{12t}$, and again you want to solve for *t*. As before, you start by dividing both sides by 5000: $\dfrac{8000}{5000} = \left(1 + \dfrac{0.055}{12}\right)^{12t}$. To get rid of the exponential, you take the natural logarithm of both sides. Using one of the properties of logarithms, you get $\ln\dfrac{8000}{5000} = 12t\ln\left(1 + \dfrac{0.055}{12}\right)$. To solve for *t*, you divide both sides by $12\ln\left(1 + \dfrac{.055}{12}\right)$ to get

$$t = \frac{\ln\frac{8000}{5000}}{12\ln\left(1 + \dfrac{0.055}{12}\right)}.$$

Again, you can punch this into the calculator as you see it, but this time you have to remember to enclose the denominator in parentheses:

ln (8000 ÷ 5000) ÷ (12 × ln (1 + 0.055 ÷ 12)) = ,

giving **8.56589097** , or about 8.57 years. The answer isn't all that much different from the answer using continuous compounding, but the calculation using continuous compounding is much simpler. One more comment: In this problem, you could use logarithms to the base 10, replacing the **ln** key by **log** (that is, $\log_{10}$), and you would get exactly the same answer. (Try it!) But in the preceding problem you need to use the **ln** key. In general, any problem that can be done using logarithms and exponentials to the base 10 can also be done, in the same way, using logarithms and exponentials to the base *e*, but some problems can be done better and more easily using logarithms and exponentials to the base *e*. I do not know of any problems in science, engineering, or business which are better done using base 10. So I suggest that you get in the habit of using only the **ln** and **e^x** keys, instead of the **log** and **10^x** keys.

Example: Finding *t* using the Present Value Formula

The following problem involves a complicated calculation, but the result is worth seeing.

Problem: (from real life!) Recently I had a credit card balance of $3130.90. My credit card company told me that I could make a minimum payment of $62.00, with an annual interest rate of 18.5% (that's right!) compounded monthly. How long would it take for me to pay off this balance with a monthly payment of $62.00, assuming I made no more charges to my credit card? How much interest would I end up paying?

Solution: The present value formula with $PV = \$3130.90$, $PMT = \$62.00$, $r = 0.185$, and $n = 12$ gives

$$3130.90 = 62.00 \frac{1 - \left(1 + \dfrac{0.185}{12}\right)^{-12t}}{\dfrac{0.185}{12}}.$$

You want to solve for t. Dividing by 62.00 and multiplying by 0.185/12, you get

$$\frac{3130.90 \times 0.185}{62.00 \times 12} = 1 - \left(1 + \frac{0.185}{12}\right)^{-12t}$$

Transposing terms, you get

$$\left(1 + \frac{0.185}{12}\right)^{-12t} = 1 - \frac{3130.90 \times 0.185}{62.00 \times 12}.$$

Taking logarithms of both sides, you get

$$-12t \ln\left(1 + \frac{0.185}{12}\right) = \ln\left(1 - \frac{3130.90 \times 0.185}{62.00 \times 12}\right).$$

To solve for t, divide both sides by $-12\ln\left(1 + \dfrac{0.185}{12}\right)$ to get:

$$t = -\frac{\ln\left(1 - \dfrac{3130.90 \times 0.185}{62.00 \times 12}\right)}{12\ln\left(1 + \dfrac{0.185}{12}\right)}.$$

You can punch this in as you see it, with parentheses around the denominator:

± ln (1 - (3130.90 × 0.185) ÷ (62 × 12)) ÷ (12 × ln (1 + 0.185 ÷ 12)) ,

giving ***8.210802828*** , or almost 8 years and 3 months (8.25 years). In 8 years and 3 months, or 99 months, at $62.00 per month, I would pay 99 × $62.00 = $6138.00 in all, including $3007.10 interest.

       I paid off my balance in full, and took a trip with the money I saved.

Thanks for the Memories

Memory Locations

       Your calculator has memory locations in which values can be stored, recalled for display, or used in expressions. These memory locations are named **A**, **B**, **C**, **D**, **E**, **F**, **M**, **X**, and **Y**. You will find **A** through **F** above and to the right of each key in the second row of black keys; you will find **X**, **Y**, and **M** to the upper right of the fourth, fifth, and sixth of the third row of black keys. You use these memory locations to save yourself the trouble of punching in the same data value more than once, copying down intermediate results on paper, and then re-entering them into an expression. Along with this, you avoid the risk of punching in, copying down, or re-entering these values incorrectly.

       To store a value appearing in the display into a memory location, you press **STO** followed by the key with the name of the memory location. To recall it into the display, you press **RCL** followed by the memory location. To use the value contained in a memory location in an expression, you press **ALPHA** followed by the memory location. The **ALPHA** key is labeled in green, and functions accessed by pressing **ALPHA** first also appear in green.

To clear a memory location, you simply store the value 0 into it. For example, to clear memory location **C**, you press **0 STO C**.

The values stored in the memory locations remain unchanged when you press **ON/C**, or even when you turn the calculator off. This is fortunate, because the calculator automatically turns itself off after five minutes or so, to save the battery, if you have not pressed a key. All of the memory locations, except for **M**, are cleared when you press **2nd F MODE** (that is, **CA**). All of the memory locations, including **M**, are cleared when you press **2nd F ALPHA** (that is, **M-CLR**) **0 =** . You can find more information about clearing your calculator in the section "**Modes and Setup**" below.

Example: Solving Quadratic Equations using Memory Locations

In the section "**Powers and Roots**" above, you used the calculator to find the solutions of quadratic equations. When the coefficients of the quadratic equation have many significant digits, you can make things easier by using the memory locations.

Example: Find the roots of the quadratic equation $ax^2 + bx + c = 0$ when a = 2.345, b = 6.543, and c = -9.876.

Solution: In the formula for the roots of a quadratic equation, each of the coefficients a, b, and c occurs twice. Rather than punch each one in twice, save them in memory locations **A**, **B**, and **C**. Then you can enter the expressions for the roots in terms of **A**, **B**, and **C**. So you punch in

2.345 STO A , 6.543 STO B , ± 9.876 STO C

If you want to check that your values have been stored properly, press **RCL A** , **RCL B** , **RCL C** . To find one of the roots, punch in

(± ALPHA B + √ (ALPHA B x^2 - 4 ALPHA A ALPHA C)) ÷ 2 ALPHA A =

to get **1.086396861** in the display. Note that you do not need to punch in the multiplication sign, nor the parentheses around the denominator, when your multiplications involve memory locations. (You still need to put parentheses around the numerator, however.) To get the other root, change the **+** to a **-** (see the section "**Editing Expressions**" above) to get **-3.876588759** in the display. Thus, the roots, to four significant figures, are 1.086 and -3.877.

More examples of the use of memory locations are given in the section "**Solving Triangles using Memory Locations**" below.

M is for Memory

You store values into memory location **M**, and you recall values from it, in just the same way as you do with the other memory locations. However, memory location **M** has some special features. When **M** contains a non-zero value, a little **m** appears at the top right corner of the display. It will disappear if you clear **M**, that is, if you press **0 STO M** . Furthermore, there is a special key affecting **M**. At the right end of the last row of black keys, you will see a key marked **M+**, and above it, in orange or red, **M-**. This key enables you to add the value in the display to the value stored in **M**, or, if you press **2nd F M+**, to subtract the value in the display from the value stored in **M**.

These features go back to the early days of calculators. In those days, there was only one memory location, named, not surprisingly, **M**. There were no parenthesis keys, and operations were not performed using the conventional order of operations, so that if you pressed **3 + 4 × 5 =** , you got **35**. But there were keys marked **STO** and **RCL**, which stored values to, and

recalled them from, the one and only memory location, namely, **M**. If you wanted to calculate, for example, $(3 \times 4) + (5 \times 6) + (7 \times 8)$, you pressed

$$3 \times 4 = \textbf{STO } 5 \times 6 = \textbf{M+ } 7 \times 8 = \textbf{M+ RCL }.$$

The **STO** operation stored the result of 3×4 into **M**, the first **M+** operation added the result of 5×6 to the result of 3×4 in **M**, and the second **M+** operation added the result of 7×8 to that. You no longer have to go through these gymnastics, but the **M+** key is still convenient in certain applications.

Example: Riemann Sums and Approximate Integration

In the old days, teachers and students rushed through the sections of the textbook on Riemann sums and approximate integration as quickly as possible, because the calculations were extremely unpleasant to do by hand. Now, however, calculators greatly simplify the work, and it is worthwhile to spend some time on this topic, both to have a clearer understanding of the basic idea of the integral, and also to learn how to get good numerical approximations to the value of an integral when the Fundamental Theorem of Calculus cannot be applied. Textbook writers have not, by and large, caught up with the advances in technology, and so we present a fairly detailed discussion of how you can use your calculator effectively for these purposes.

We take as our basic example the integral $\int_3^6 \sqrt{x-2}\,dx$. You can easily evaluate this integral using the Fundamental Theorem of Calculus, obtaining a value of $14/3 = 4.666666\ldots$. We break the interval $[3,6]$ into $n = 6$ subdivisions, each of width $\Delta x = (6 - 3)/6 = 0.5$. The left-hand endpoint of the interval of integration is 3.0, and the right-hand endpoint is 6.0. The intermediate subdivision points are at 3.5, 4.0, 4.5, 5.0, and 5.5. It will be convenient to write the endpoints and intermediate points as 6/2, 7/2, 8/2, 9/2, 10/2, 11/2, and 12/2. We recall that a Riemann sum is a sum of the form $\sum_{i=0}^{n} f(x_i)\,\Delta x$, where x_i is an x-value in the i-th subinterval.

You start by storing Δx in memory location **D**:

$$\textbf{0.5 STO D }.$$

This value will stay unchanged throughout the following examples.

Example: Find the left sum of $\int_3^6 \sqrt{x-2}\,dx$ with $n = 6$.

Solution: First clear **M** (don't forget this step!):

$$\textbf{0 STO M }.$$

In this case, the values of x_i are the left-hand endpoints of the subintervals, namely, 6/2, 7/2, 8/2, 9/2, 10/2, and 11/2. Punch in

$$\sqrt{} \textbf{ (6 } \div \textbf{ 2 - 2) } \times \boxed{\textbf{ALPHA}} \textbf{ D M+}$$

to get, not surprisingly, **0.5** in the display. Now, press the right arrow key ($\Rightarrow$) three times, so that it is over the **6**, and replace the **6** by a **7**. (See the section "**Editing Expressions**" to see how this is done on your calculator.) Then press **M+** to get **0.612372435** in the display. This is the value of $\sqrt{7/2 - 2} \times 0.5$, and it has been added to the value in **M**. Similarly, replace the **7** by **8** and press **M+**, the **8** by **9** and press **M+**, the **9** by **10** (careful here!) and press **M+**, and finally the **10** by **11** and press **M+**. Now recall the value in **M**:

$$\textbf{RCL M}$$

and if all has gone well, you should see **4.411488382** in the display. So the left sum for $n = 6$ is 4.411488382 – not very close to 4.666… . Save this value in memory location **E** for future reference by pressing **STO E** .

   Example: Find the right sum of $\int_3^6 \sqrt{x-2}\ dx$ with $n = 6$.

   Solution: Again, clear **M**: **0 STO M** . This time, the values of x_i are the right-hand endpoints of the subintervals, namely 7/2, 8/2, 9/2, 10/2, 11/2, and 12/2. Punch in
         √ (7 ÷ 2 - 2) × ALPHA D M+
to get **0.612372435** in your display. As before, replace the **7** by **8** and press **M+**, the **8** by **9** and press **M+**, the **9** by **10** and press **M+**, the **10** by **11** and press **M+**, and the **11** by **12** and press **M+**. Now recall the value in **M**, and if all has gone well, you should have **4.91188382** in your display. This is your right sum. Save it in memory location **F** for future reference.

   Example: Find the trapezoidal sum of $\int_3^6 \sqrt{x-2}\ dx$ with $n = 6$.

   Solution: Since $f(x) = \sqrt{x-2}$ is increasing on [3,6], the left sum lies entirely under the graph of the function, and the right sum lies above the graph. So a reasonable thing to do is to take their average. You can do this by punching in
         (ALPHA E + ALPHA F) ÷ 2 = ,
giving **4.661488382**. Geometrically, this corresponds to the sum of the areas of the trapezoids formed by connecting the points on the graph corresponding to x = 6/2, 7/2, … , 12/2. For this reason, this method of estimating the value of the integral is called the Trapezoid Rule. Save it in memory location **A** for future reference.

   You can calculate the trapezoidal sum directly, without calculating the left and right sums first. Start by calculating the average of $f(3)\Delta x$ and $f(6)\Delta x$, and store the result in M:
   (√ (3 - 2) × ALPHA D + √ (6 - 2) × ALPHA D) ÷ 2 STO M
giving **0.75**. (You could have calculated this by hand, but why bother? The important thing is that it gets stored in **M** to start the process.) Now you punch in $f(x_i)\Delta x$ for the intermediate subdivision points (not including the endpoints), that is, x_i = 7/2, 8/2, 9/2, 10/2, and 11/2:
         √ (7 ÷ 2 - 2) × ALPHA D M+ .
As before, replace the **7** by **8** and press **M+**, the **8** by **9** and press **M+**, the **9** by **10** and press **M+**, the **10** by **11** and press **M+**. Stop here! Now recall the value in **M**, which should give you **4.661488382**, the same as before. Save it in memory location **A** for future reference. This is a respectable approximation to 4.666… , but we can do better.

   Example: Find the midpoint sum of $\int_3^6 \sqrt{x-2}\ dx$ with $n = 6$.

   Solution: To do this, take the points x_i to be the midpoints of each of the subintervals, namely 13/4, 15/4, 17/4, 19/4, 21/4, and 23/4, and repeat the process described above. You start, of course, with **0 STO M** , and continue by punching in
         √ (13 ÷ 4 - 2) × ALPHA D M+ .
Then replace the **13** by **15** and press **M+**, the **15** by **17** and press **M+**, the **17** by **19** and press **M+**, the **19** by **21** and press **M+**, and finally the **21** by **23** and press **M+**. Now recall the value in **M**, which should be **4.669244675**. Store this value in memory location **B** for further reference. Note that this also is a respectable approximation to 4.666… , in fact, even closer than the approximation given by the Trapezoidal Rule.

   In the old days, textbooks passed over the midpoint sum, because it generally involved messier numbers (13/4, 15/4, … instead of 6/2, 7/2, …) than the Trapezoidal Rule. Nowadays,

the fact that the numbers are messier is no obstacle to your calculator. Calculating the midpoint sum is no harder that applying the Trapezoidal Rule, in fact, it is actually somewhat easier. If you go into the mathematical theory, you discover that in general the difference between the value given by the midpoint sum and the true value of the integral is about half the difference between the value given by the Trapezoidal Rule and the true value of the integral. Furthermore, when the graph of the function is concave down, as it is for $f(x) = \int_3^6 \sqrt{x-2}\ dx$, the Trapezoidal Rule gives a value less than the exact value of the integral, and the midpoint sum gives a value greater than the exact value of the integral. When the graph of the function is concave up, all the inequalities are reversed. So you can get an even better approximation to the exact value of the integral by taking a weighted average of the values given by the Trapezoidal Rule and the midpoint sum, giving the midpoint sum twice as much weight as the trapezoidal sum. This method for approximating integrals is called Simpson's Rule.

   Example: Approximate $\int_3^6 \sqrt{x-2}\ dx$ with $n = 6$ using Simpson's Rule.

   Solution: Since you stored the results from the Trapezoidal Rule and the midpoint sum in memory locations **A** and **B** respectively, you simply punch in

   (ALPHA A + 2 ALPHA B) ÷ 3 =

to get **4.666659244**. This is an excellent approximation to the true value! This is typical of results obtained by Simpson's Rule. In fact, Simpson's Rule is an industrial-strength method for approximating the values of definite integrals widely used by engineers and applied mathematicians when it is difficult or impossible to find an anti-derivative. An example of this would be $\int_3^6 \sqrt{x^3 - 2}\ dx$.

   Note: What we have given above as the Simpson's Rule approximation for the integral with $n = 6$ is what some books would call the approximation for $n = 12$. What they call the approximation for $n = 6$ is the Simpson's Rule approximation obtained by our method starting from the Trapezoidal Rule and midpoint sum for $n = 3$.

   To make sure you understand how to use the above method, try it on $\int_0^1 \dfrac{4}{1+x^2}\ dx$ with

$n = 4$. The true value of this integral is $\pi = 3.141592654\ldots$. The values given by the left sum, right sum, Trapezoidal Rule, midpoint sum, and Simpson's Rule are respectively 3.381176471, 2.881176471, 3.131176471, 3.146800518, and 3.141592502.

Play it again, Sam

   Your calculator has a feature, called "playback", which enables you (sometimes) to see previous expressions and their results. It is accessed using the up arrow (⇑) and down arrow (⇓) keys at the top of the keyboard. To see how it works, press

   3 + 4 × 5 =

to get **23.** , and

   6 + 7 × 8 =

to get **62.** in the display. Press the ⇑ key. You will see

   3 + 4 × 5 =
                    23.

again, with a small triangle pointing downwards (   ) at the left of the top line of the display. If you now press ⇓ , you will see

$$6 \ + \ 7 \ \times \ 8 \ =$$
$$62.$$

once again, and there will be a small triangle pointing upwards (   ). You can flip back and forth between the expressions using ⇑ and ⇓. You can enter more expressions and flip between them. When you are at the "bottom" (most recent) expression, you will see only the triangle pointing upwards; when you are at the "top" (oldest) expression, you will see only the triangle pointing downwards; and when you are between the two, you will see both triangles.

There are certain limitations on what you can do. The first is the limitation of memory. The calculator can hold a maximum of 142 characters. When this memory is full, expressions are deleted, starting with the oldest. The second is that under certain circumstances, this memory is cleared. Not surprisingly, it is cleared by the various mode and reset operations (see "**Modes and Setup**" below). Unlike the memory locations **A**, **B**, **C**, **D**, **E**, **F**, **M**, **X**, and **Y**, it is also cleared when the calculator is turned off (including when the calculator turns itself off automatically). It is not cleared when you press **ON/C** .

On the Sharp, it is rather annoyingly cleared when you store a value to any of the memory locations. In the example above where we found the solutions of a quadratic equation by storing the coefficients a, b, c into memory locations **A**, **B**, **C** respectively, it would be nice to be able to change the values of these coefficients and find solutions of the resulting equation without typing the quadratic formula back into the calculator.

Playing the Angles

In this section, we will assume, unless otherwise stated, that your calculator is set for degrees, with "**DEG**" appearing at the top. If this is not the case, see the section "**Degrees, Radians, Grads**" for your calculator under "**Modes and Setup**" below. We will also assume that we are dealing with a triangle ABC in which a, b, c are the lengths of the sides opposite the angles A, B, and C respectively.

Right Triangles

In this section, we will assume that the triangle ABC is a right triangle, with right angle at C. Then we have the relations $c^2 = a^2 + b^2$, $\sin A = \cos B = \dfrac{a}{c}$, $\cos A = \sin A = \dfrac{b}{c}$, $\tan A = \dfrac{a}{b}$, $\tan B = \dfrac{b}{a}$. The following examples illustrate the use of the calculator in solving for various parts of the triangle.

Example: Find c, given a = 3.45 m and b = 7.65 m.

Solution: In this case, you have $c = \sqrt{a^2 + b^2}$, so you press
$$\surd \ (\ 3 \ . \ 4 \ 5 \ x^2 \ + \ 7 \ . \ 6 \ 5 \ x^2 \ ) \ =$$
to get **8.391960438** . (Note the parentheses!) Rounding off to three significant figures (because the data are given with three significant figures), our answer is c = 8.39 m.

Example: Find a, given c = 4.567 cm and B = 34.56°.

Solution: In this case, you have $a = c \cos B$, so you press
$$4.567 \ \times \ \cos \ 34.56 \ =$$
to get **3.761073373** . Rounding off to four significant figures, our answer is a = 3.761 cm.

Example: Find A, given a = 3.45 m and b = 7.65 m.

Solution: In this case, you have $\tan A = \dfrac{a}{b}$. In order to solve for A, you have to "undo" the tan function. This is done by the tan^{-1} function, sometimes called the arctan function. To access this function, press **2nd F tan** . Refer back to the section "**Keys for Functions and their Inverses**" under "**Exponentials and Logarithms**" above. As indicated in that section, we will simply write **tan^{-1}** for **2nd F tan** ; similarly for **sin^{-1}** and **cos^{-1}**. So to find A, you press

tan^{-1} (3.45 ÷ 7.65) =

to get **24.27444113** . (Again, note the parentheses!) Rounding off to three significant figures, our answer is $A = 24.3°$.

These examples can also be done using the procedures described in the section "***Converting between Rectangular and Polar Coordinates***" below.

Degrees, Minutes, Seconds

Your calculator can handle calculations involving degrees, minutes, and seconds. These are handled by the key labeled **D°M′S**, which is the third key from the left in the third row of black keys. To enter an angle expressed in degrees, minutes, and seconds, press this key to separate the minutes from the degrees and the seconds from the minutes.

Example: In a right-angled triangle ABC, with right angle at C, find a, given $c = 4.567$ cm and $B = 12°34′56″$.

Solution: As above, you have $a = c \cos B$, so you press

4.567 × cos 12 D°M′S 34 D°M′S 56 =

to get **4.457320758**, which you round off to four significant figures as 4.457 cm.

To convert an angle expressed as degrees, minutes, and seconds into an angle expressed as degrees and decimals or vice versa, you use **2nd F D°M′S**.

You can also use the **D°M′S** key to convert an answer from degrees and decimal parts to degrees, minutes, and seconds.

Example: Find A to the nearest minute, given $a = 3.456$ m and $b = 7.654$ m.

Solution: As above, you have $\tan A = \dfrac{a}{b}$. So you punch in

tan^{-1} (3.456 ÷ 7.654) =

to get **24.30053777**. You then press **2nd F D°M′S** (that is, ↔DEG) to get **24°18′1.936″**. So the answer, to the nearest minute, is $A = 24°18′$.

Actually, your calculator does not care whether you are thinking of degrees and minutes or of hours and minutes, as the next remarkable example shows:

Example: One day you worked from 1:43 pm to 5:21 pm. How long did you work, and how much did you earn, at $12.34 per hour?

Solution: To find out how long you worked, subtract 1:43 from 5:21 by pressing

5 D°M′S 21 - 1 D°M′S 43 =

giving **3°38′0″**, that is 3 hours and 38 minutes. Then multiply this by 12.34 by pressing

× 12.34 =

to get **44°50′7.2″** (in hours, minutes, and seconds!). Convert this to decimals by pressing ↔**DEG** to get **44.83533333**. Thus, you earned $44.84 (assuming your employer rounds your pay up). Notice that your calculator does arithmetic (correctly!) on numbers expressed in degrees, minutes, and seconds.

Law of Cosines

In the old days (before 1980!), the Law of Sines was taken up first and studied in detail, followed by the Law of Cosines, and there was a tacit understanding between teachers and students that examination questions on the Law of Cosines would require only simple calculations. The reason for this was that calculations involving the Law of Sines could be done purely by logarithms (see the section "**Why e? Why Natural Logarithms?**" above), while calculations involving the Law of Cosines generally required a messy combination of hand calculation and use of logarithms unless the numbers were carefully chosen. Actually, though, the Law of Cosines is more important for applications than the Law of Sines, because of its connection with the dot (scalar) product in vector geometry. Furthermore, with the calculator, it is not significantly harder to use than the Law of Sines.

You recall that for a triangle ABC, with sides a, b, c and angles A, B, C, where we no longer assume that angle C is a right angle, the Law of Cosines can be expressed as

$$c^2 = a^2 + b^2 - 2ab \cos C \qquad \text{or as} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

with corresponding expressions for the other sides and angles.

Example: Find c, given a = 3.456 m, b = 7.654 m, and C = 23°45′.

Solution: Press

$\sqrt{}$ **(3.456 x^2 + 7.654 x^2 - 2 × 3.456 × cos 23 D°M′S 45) =**

to get **4.701446953**. To four significant figures, the answer is 4.701 m.

Example: Find C to the nearest minute, given a = 2.345 cm, b = 3.456 cm, and c = 5.432 cm.

Solution: Press

cos^{-1} ((2.345 x^2 + 3.456 x^2 - 5.432 x^2) ÷ (2 × 2.345 × 3.456)) =

to get **138.0967454**. Then press **↔DEG** to get **138°5′48.28″**. Rounded off to the nearest minute, the answer is 138°6′. Notice that the calculator automatically gives an obtuse angle, because cos C turns out to be negative.

Law of Sines

The Law of Sines can be expressed as $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

Example: Find a if A = 34°56′, B = 54°32′, and c = 76.54 m.

Solution: Solving for a in terms of c, A, and C, you have $a = \dfrac{c \sin A}{\sin C}$. You are not given C, but you can find it, since C = 180° - (A + B). Actually, you do not need to calculate it separately. You can do it all in one calculation, without converting the degrees and minutes to decimals:

76.54 × sin 34 D°M′S 56 ÷ sin (180 - (34 D°M′S 56 + 54 D°M′S 32)) = ,

giving **43.83045729**. To four significant figures, the answer is 43.83 m.

Do not use the Law of Sines to solve for angles of a triangle, because the Law of Sines cannot distinguish between an acute angle and an obtuse angle. See the discussion of this matter in the section "**Solving Triangles using Memory Locations**" below.

Converting between Degrees and Radians

Sometimes you need to convert an angle given in degrees to the same angle given in radians. Of course you can always multiply by $\pi/180$ or $180/\pi$, if you can remember which one to use, but your calculator can make the conversion for you.

To convert from an angle in degrees to an angle in radians, start with your calculator in **DEG** mode. (If necessary, press the **DRG** key until you see **DEG** in small letters at the top of your display.) Enter your value into the display. Then press **2nd F** π (that is, **DRG**), not to be confused with the **DRG** key just mentioned, nor the **↔DEG** key above the **D°M′S** key. Notice that the small **DEG** will change to **RAD**, that is, your calculator will change into **RAD** mode.

   Example: Convert 60° to radians.
   Solution: In **DEG** mode, press **60 2nd F** π. The display will read

RAD

   60 RAD
   1.047197551

Thus, 60° equals approximately 1.047 radians. Note that your calculator is now in **RAD** mode.

To convert from an angle in radians to an angle in degrees, start with your calculator in **RAD** mode. Enter your value into the display. Then press **DRG** twice. Note that your calculator is now in **DEG** mode.

   Example: Convert 2.5 radians to degrees.
   Solution: Make sure you are in **RAD** mode. Press **2.5 DRG DRG** . The display will read

DEG

   ANS DEG
   143.2394488

Thus, 2.5 radians is approximately 143.24°. Note that your calculator is now in **DEG** mode.

Solving Triangles Using Memory Locations

Given a triangle ABC, we will use memory locations **A**, **B**, and **C** to store the lengths of the three sides a, b, and c, and the memory locations **D**, **E**, and **F** to store the sizes of the three angles A, B, and C respectively. Given three of the six quantities a, b, c, A, B, C, to "solve the triangle ABC" means to find (if possible) the remaining three.

Given Three Sides ("SSS Case")

   Example: Solve the triangle ABC, given $a = 5.678$ m, $b = 6.543$ m, $c = 7.654$ m.
   Solution: First store the values of a, b, and c in memory locations **A**, **B**, and **C** respectively:
   5.678 STO A ,   6.543 STO B ,   7.654 STO C
Now use the Law of Cosines to solve for A, B, and C. Recall that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac}, \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

So for angle A, you punch in
cos^{-1} ((ALPHA B x^2 + ALPHA C x^2 - ALPHA A x^2) ÷ 2 ALPHA B ALPHA C) STO D
to get *46.33485975*. Note that you can press **STO D** at the end of your expression to have the value stored in the memory location as soon as it is calculated. Similarly, you calculate angle B and angle C, and store the values in memory locations **E** and **F** as follows:

cos^{-1} ((**ALPHA** A x^2 + **ALPHA** C x^2 - **ALPHA** B x^2) ÷ 2 **ALPHA** A **ALPHA** C) STO E
giving ***56.46929364***, and
cos^{-1} ((**ALPHA** A x^2 + **ALPHA** B x^2 - **ALPHA** C x^2) ÷ 2 **ALPHA** A **ALPHA** B) STO F
giving ***77.19584662***. You can check this by seeing that the sum of the angles is 180°:
                   ALPHA D + **ALPHA** E + **ALPHA** F =
giving exactly ***180***. So to the nearest minute, the angles are $A = 46°20′$, $B = 56°28′$, $C = 77°12′$.

Given Two Sides and the Included Angle ("SAS Case")

Example: Solve the triangle ABC, given $a = 3.456$ m, $b = 7.654$ m, and $C = 23°45′$.
Solution: Store the values for a, b, and C in memory locations **A**, **B**, and **F** respectively.
As in the example in the section "***Law of Cosines***" above, you get ***4.701446953***, which you
save in memory location **C**. The usual advice at this point is to use the Law of Sines to find
angles A and B. Thus, $\sin A = \dfrac{a \sin C}{c}$ and $\sin B = \dfrac{b \sin C}{c}$. So, punching in
          sin^{-1} (**ALPHA** A sin **ALPHA** F ÷ **ALPHA** F) = STO D ,
we get ***17.22088307***, and punching in
          sin^{-1} (**ALPHA** B sin **ALPHA** F ÷ **ALPHA** F) = STO E ,
we get ***40.97088307***. Unfortunately, the sum of **D**, **E**, and **F** is nowhere near 180°. What has
gone wrong? We try again, using the Law of Cosines. Punching in
          cos^{-1} ((**ALPHA** B x^2 + **ALPHA** C x^2 - **ALPHA** A x^2) ÷ 2 **ALPHA** B **ALPHA** C) ,
we get ***17.22088307*** as before, but punching in
          cos^{-1} ((**ALPHA** A x^2 + **ALPHA** C x^2 - **ALPHA** B x^2) ÷ 2 **ALPHA** A **ALPHA** C) ,
we get ***139.0291169***, which we store in **E**. This time the sum of **D**, **E**, and **F** is 180°. The
explanation is that angle B is an obtuse angle, whose cosine is negative. The angles 40.97…°
and 139.029…° are supplementary (add up to 180°), and their sines are equal. The inverse sine
function sin^{-1} gives us only the acute angle, but the inverse cosine function cos^{-1} distinguishes
between the acute and the obtuse angle. The moral of the story: **DO NOT USE THE LAW OF
SINES TO CALCULATE ANGLES!** The answers to the problem are $c = 4.701$ m,
$A = 17°13′$, $B = 139°2′$.

Given Two Angles and the Included Side ("ASA Case")

Example: Solve the triangle ABC, given $A = 34°45′$, $B = 76°54′$, $c = 2.345$ m.
Solution: Store the values for A, B, and c in memory locations **D**, **E**, and **C** respectively.
You can find angle C from the fact that the sum of the angles is 180°:
          180 - (**ALPHA** D + **ALPHA** E) STO F ,
so that angle $C = 68°21′$. Now you can find (correctly!) side a and side b using the Law of
Sines:
          ALPHA C sin **ALPHA** D ÷ sin **ALPHA** F STO A
gives ***1.438092836***, and
          ALPHA C sin **ALPHA** E ÷ sin **ALPHA** F STO B
gives ***2.457325994***. You can check this by calculating c, using the Law of Cosines and your
calculated values for a and b:
          √ (**ALPHA** A x^2 + **ALPHA** B x^2 - 2 **ALPHA** A **ALPHA** B cos F) =
gives ***2.345***, as it should. So the answers are $C = 68°21′$, $a = 1.438$ m, $b = 2.457$ m.

Given Two Angles and a Side Opposite ("AAS Case")

Given two angles, you can find the third, using the fact that the sum of the angles is 180°. You can then find the remaining sides, using the Law of Sines as in the ASA Case.

Given Two Sides and an Angle Opposite ("ASS Case")

Example: Solve the triangle ABC, given a = 4.321 m, b = 5.432 m, and A = 34°56′.

Solution: Store the values for a, b, and A in memory locations **A**, **B**, and **D** respectively. The case of the Law of Cosines involving sides a and b and angle A is
$$a^2 = b^2 + c^2 - 2bc \cos A \ .$$
We are given a, b, and A; we want to solve this equation for c. We can rewrite the equation as
$$0 = c^2 - 2bc \cos A + (b^2 - a^2),$$
and think of it as a quadratic equation for the unknown c, where the coefficient of c^2 is 1, the coefficient of c is $-2b \cos A$, and the constant term is $b^2 - a^2$. Using the quadratic formula, we find
$$c = b \cos A \pm \sqrt{a^2 - (b \sin A)^2} \ .$$
(This formula is easy to interpret geometrically.) There will be no solutions if $a < b \sin A$, there will be exactly one positive solution if $a = b \sin A$, there will be two positive solutions if $b \sin A < a < b$, and there will be only one positive solution if $a \geq b$. Since c represents a length, only positive solutions are geometrically meaningful. In this case, you can punch in
ALPHA B cos ALPHA D + √ (ALPHA A x^2 - (ALPHA B sin ALPHA D) x^2) =
to get **7.452573925**, which you can store in **C**. Changing the **+** sign to **-**, you get **1.453938345**, which you can temporarily store in, for example, **M**. Now you use the Law of Cosines, as in the SSS and SAS cases above, to get **46.04240052** for angle **B**, which you store in **E**, and **99.02426614** for **C**, which you store in **F**. Next, you fetch the second value for c, which you stored in **M**, and move it to **C**: **RCL M STO C** . You can then calculate the angles **B** and **C** as before, getting **133.9575995** and **11.10906719** respectively, or you can observe that the second value for angle **B** is the supplement of the first, that is, 180° - 46.042…°, and that the second value for angle **C** can be obtained from the fact that the sum of the angles of the triangle must be 180°. So, in either case, the answers are c = 7.453, **B** = 46°3′, **C** = 99°1′, and c = 1.454, **B** = 133°57′, **C** = 11°7′. (Notice that in both triangles there is an obtuse angle, so that the Law of Sines does not suffice for the calculation.)

Rectangular and Polar Coordinates

Your calculator can convert coordinates of points in the plane given in rectangular (xy) coordinates) into polar ($r\theta$) coordinates. Observe the orange comma (**,**) above and to the left of the **STO** key, the orange symbols ←,→ above the Exp key at the left of the last row of black keys, the →$r\theta$ above the 8 key, and the →xy above the 9 key. These are used in converting between rectangular and polar coordinates. The results of the calculation are stored in memory locations **X** and **Y**. The following examples show how this is done.

Example: Convert the rectangular coordinates (3,-4) to polar coordinates.

Solution: Press **3 2nd F STO ± 4 2nd F 8** , that is, **3 , ±4 →$r\theta$** . You will see
r=
                    5.
To find the θ value, press **2nd F Exp** , that is, ←,→ . You will now see

> **θ=**
> ***-53.13010235***

You can flip back to the *r*-value by pressing **2nd F Exp** , that is, ←,→ again. You now see

> **r=**
> ***5.***

The *r*-value and the *θ*-value can be accessed by pressing **RCL X** and **RCL Y** . So the polar coordinates of the point (3,-4) are (5,-53.13°).

   Example: Convert the polar coordinates (8,-60°) to rectangular coordinates.

   Solution: Press **8 2nd F STO ± 60 2nd F 9** , that is, **8 , ±60 →xy** . You will see

> **x=**
> ***4.***

To find the *y* value, press **2nd F Exp** , that is, ←,→ . You will now see

> **y=**
> ***-6.92820323***

You can flip back to the *x*-value by pressing **2nd F Exp** , that is, ←,→ again. You now see

> **x=**
> ***4.***

The *x*-value and the *y*-value are stored in memory locations **X** and **Y**, and can be accessed by pressing **RCL X** and **RCL Y** . So the rectangular coordinates of the point (8,-60°) are (4,-6.928).

Sinh, Cosh, and All the Hype

   To the left of the **sin**, **cos**, and **tan** keys, you find a key labeled **hyp**. This key gives you access to the so-called hyperbolic functions sinh, cosh, and tanh (pronounced "sinch", "cosh", and "tanch"), which play an important role in some kinds of physics and engineering problems.   Actually, these functions are really nothing new; they are defined, using the exponential function, as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \ \cosh x = \frac{e^x + e^{-x}}{2}, \ \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Their names derive from the fact that they have properties analogous to, but often slightly different from, the trigonometric functions. For example, it is easy to see that

$$\tanh x = \frac{\sinh x}{\cosh x}.$$

On the other hand, it is not immediately obvious, but you can easily check, that $\cosh^2 x - \sinh^2 x = 1$ (instead of $\cos^2 x + \sin^2 x = 1$). If you are one of the relatively small number of people who need to use hyperbolic functions, you probably need to use them a lot. So it is convenient to be able to calculate sinh 1.234, for example, by punching in

> **hyp sin 1.234 =**

to get ***1.571908059***, and similarly for the other hyperbolic functions. You do not need to worry about degrees or radians; the hyperbolic functions are completely independent of the degree-radian-grad setting.

   The inverses of the hyperbolic functions are also important. Just as the hyperbolic functions are expressed in terms of the exponential function, their inverses can be expressed in terms of the ln function:

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}), \quad \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), \quad \tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}.$$

You obtain these, not surprisingly, using the **2nd F** key. To get cosh^{-1} 2.345, for example, you can punch in either

> **hyp 2nd F cos 2.345 =**

or

> **2nd F hyp cos 2.345 =**

to get **1.496513422** . What happens if you punch in

> **2nd F hyp 2nd F cos 2.345 =** ?

Does this seem reasonable?

Fractions, Proper and Improper

In the old days, we used to spend a lot of time working with fractions. There were 12 inches to a foot, 3 feet in a yard, 16 ounces in a pound, … , and all sorts of conversions between them. Now we have the metric system, where almost everything can be handled by decimals, and as a result you have probably had less contact with operations such as 2/3 + 3/4. Fortunately, your calculator can take care of such things for you. Your calculator has a key marked **a b/c**, with **d/c** in orange above, and this key is used for calculations with fractions. The **a b/c** key is the second from the left in the third row of black keys.

You can enter a fraction either as a simple ratio, such as 2/3, or as an integer followed by a ratio, such as 2 3/4. We look first at the case of simple ratios. To calculate 2/3 + 4/5, you press

> **2 a b/c 3 + 4 a b/c 5 =** .

In your display, you will see

> **2 ⌐3+3 ⌐4=**
> **1 ⌐7 ⌐15**

A symbol ⌐ between two expressions means the fraction formed from the two expressions. Two symbols ⌐ separating three expressions means the integer (whole number) value of the first part plus the fraction formed by the second and third parts. Thus, the result in the second line of the display means 1 7/15, which is the sum of 2/3 and 4/5. You can enter values as improper fractions (where the numerator is greater than the denominator). Thus, if you press

> **5 a b/c 3 + 13 a b/c 4 =** ,

you will see

> **5 ⌐3+13 ⌐4=**
> **4 ⌐11 ⌐12**

meaning that 5/3 + 13/4 = 4 11/12. If you want to enter a fraction as an integer followed by a fractional part, you press **a b/c** between the integer and the numerator, and again between the numerator and the denominator. Thus, to repeat the above calculation as 1 2/3 + 3 1/4, you press

> **1 a b/c 2 a b/c 3 + 3 a b/c 1 a b/c 4 =** ,

and again you get **4 ⌐11 ⌐12** , that is, 4 11/12.

Your fractions do not have to be entered in lowest terms. If you press

> **6 a b/c 9 + 16 a b/c 20 =** ,

you will still get **1 ⌐11 ⌐15**, just as before – your calculator will reduce the results to lowest terms.

You can add, subtract, multiply, or divide these fractions, and your result will be expressed as an integer plus a proper fraction. You can also square and cube fractions, using the x^2 and x^3 keys. You can even take square roots and cube roots, providing that the result is a

rational number. If you press, for example, $\sqrt{\ }$ **(16   25) =** , you will get **4   5** as the result. If you try to perform an operation whose result is not a rational number, you will get the usual decimal approximation. You can also get the decimal approximation to a result by pressing the **a b/c** key. For example, if you press (as above) **2   a b/c   3 + 4   a b/c   5 =** , you get the result **1   7   15** . If you now press **a b/c**, you will get **1.466666667**, which is the decimal approximation to 1 7/15. If you press **a b/c** once more, the display will return to **1   7   15**. If you now press **2nd F a b/c** , that is, **d/c**, you will see the result displayed as an improper fraction, in this case **22   15**. If you press **d/c** again, the display will return to the form **a b/c**, in this case **1   7   15**. In short, you can do just about anything you like with these fractions. One warning: If, for example, you press **4   5 x^2 =** , the result is **16   25**, meaning that the calculator has interpreted your input as (**4   5) x^2 =** . If you want instead $4 \div 5^2$, that is, 4/25, you have to punch it in as **4   (5 x^2)** . The remedy, of course, is to insert parentheses to show exactly what you want.

You might like to try the complicated fraction calculation given in the section "**Getting Started**" above. If all goes well, you should get the answer -5 575/858, or -4865/858.

Example: Row Reduction of Matrices

When you take Elementary Linear Algebra, you learn how to find the solutions of systems of linear equations by row-reduction of the corresponding augmented matrix. The problem is that unless the coefficients are very carefully chosen, the calculations with fractions tend to be messy, and it is easy for errors to creep in. Sometimes the fractions can be avoided by clever interchanges of rows, but these tend to obscure the simple underlying idea of the Gauss-Jordan method. However, you can use the fraction-manipulating capabilities of your calculator to enable you to follow the straight Gauss-Jordan method without getting bogged down in messy fractions. We illustrate this idea with a 2×2 system.

Example: Solve the system
$$3x + 5y = 7$$
$$4x - 7y = 2.$$

Solution: The augmented matrix of this system is
$$\begin{bmatrix} 3 & 5 & 7 \\ 4 & -7 & 2 \end{bmatrix}.$$

We denote the rows of this matrix by $\mathbf{R}_1$ and $\mathbf{R}_2$. The first step is to get a 1 in the first column of $\mathbf{R}_1$. You can do this by dividing $\mathbf{R}_1$ by 3. It's not too much strain to do this by hand:
$$\begin{bmatrix} 1 & 5/3 & 7/3 \\ 4 & -7 & 2 \end{bmatrix}.$$

Next, you want to get a 0 in the first column of $\mathbf{R}_2$. You do this by taking $\mathbf{R}_2$ - $4 \times \mathbf{R}_1$. It is easy enough to see that this gives 0 in the first column of $\mathbf{R}_2$ as desired, but to get the remaining entries of $\mathbf{R}_2$, it is convenient to use the fraction operations of your calculator. For the second column of $\mathbf{R}_2$, press

± 7 - 4 × 5 a b/c 3 =

giving **-13 2 3**, that is, -13 2/3. Similarly, for the third column of $\mathbf{R}_2$, press

2 - 4 × 7 a b/c 3 =

giving **-7 1 3**, that is, -7 1/3. Thus, the result of the second row operation is

$$\begin{bmatrix} 1 & 5/3 & 7/3 \\ 0 & -13\,2/3 & -7\,1/3 \end{bmatrix}.$$

Now you want to get a 1 in the second column of $\mathbf{R}_2$, by dividing $\mathbf{R}_2$ by -13 2/3. You or I would have to do this by first converting -7 1/3 and -13 2/3 into the improper fractions -22/3 and -41/3, but your calculator can do the whole thing as is:

± 7 a b/c 1 a b/c 3 ÷ ± 13 a b/c 2 a b/c 3 =

giving a result of **22   41**, that is, 22/41. So this row operation gives

$$\begin{bmatrix} 1 & 5/3 & 7/3 \\ 0 & 1 & 22/41 \end{bmatrix}.$$

Finally, you take $\mathbf{R}_1$ - 5/3 × $\mathbf{R}_2$, by punching in

7 a b/c 3 - 5 a b/c 3 × 22 a b/c 41 = ,

giving **1   18   41**, that is, 1 18/41 or 59/41. So the row-reduced matrix is

$$\begin{bmatrix} 1 & 0 & 1\,18/41 \\ 0 & 1 & 22/41 \end{bmatrix},$$

and the solution of the system of equations is $x = 1\,18/41$, $y = 22/41$. It is easy to check that these values do in fact satisfy the system of equations.

It is generally not worth bothering with row-reduction for a 2×2 system. A more realistic system is the following:

Example: Solve the system

$$3x + 4y - 5z = 9$$
$$4x - 5y + 6z = -8$$
$$-5x + 6y + 7z = 7$$

Solution: We give only the augmented matrices for each step. You can use them to check your calculations.

$$\begin{bmatrix} 3 & 4 & -5 & 9 \\ 4 & -5 & 6 & -8 \\ -5 & 6 & 7 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4/3 & -5/3 & 3 \\ 0 & -10\,1/3 & 12\,2/3 & -20 \\ 0 & 12\,2/3 & -11/3 & 22 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4/3 & -5/3 & 3 \\ 0 & 1 & -17/31 & 1\,29/31 \\ 0 & 0 & 14\,6/31 & -2\,16/31 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4/3 & -5/3 & 3 \\ 0 & 1 & -17/31 & 1\,29/31 \\ 0 & 0 & 1 & -39/220 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4/3 & 0 & 2\,31/44 \\ 0 & 1 & 0 & 1\,79/110 \\ 0 & 0 & 1 & -39/220 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 91/220 \\ 0 & 1 & 0 & 1\,79/110 \\ 0 & 0 & 1 & -39/220 \end{bmatrix}$$

So the solution is $x = 91/220 = 0.413636\ldots$, $y = 1\,79/110 = 1.71818\ldots$,
$z = -39/220 = -0.177272\ldots$.

Let Me Count the Ways …

There are many situations in which you need to know the number of ways something can be done. Here are some simple ones: How many ways are there of ranking ten contestants in a contest? How many ways are there of selecting a president, a secretary, and a treasurer from a club having ten members? How many ways are there of selecting a committee of three members from the club having ten members? You can use your calculator to find the answers to these questions, and others like them.

For the first problem, you can reason as follows: There are 10 ways of choosing the winner; after that one is chosen, there are 9 contestants left, so there are 9 ways of choosing the second place; thus, there are 10×9 = 90 ways of choosing the first and second places. Continuing in this way, there are 10×9×8 = 720 ways of choosing the first, second, and third places, 10×9×8×7 = 5040 ways of choosing the first four places, and so on until you have chosen the first nine places, in which case there is only one contestant left to bring up the rear. So there are 10×9×8×7×6×5×4×3×2×1 = 3628800 ways of ranking the ten contestants. This function, where you multiply together all the positive integers starting with some positive integer n and going down to 3×2×1, is called the *factorial function*, and is denoted by $n!$. Thus, $n! = n×(n-1)×(n-2)×\ldots×3×2×1$. (You can think of the exclamation point as standing for "Wow!", because $n!$ gets big very fast as n increases.) This function is available on your calculator. You get it by pressing **2nd F** followed by the **4** key. You observe that $n!$ occurs in yellow above the key in question. Thus, to find 10!, you press

10 2nd F 4 =

to get ***3628800.*** From now on, we will simply indicate the key combination for the factorial by **!**. If you try **1 ! =**, you will get 1, which is reasonable enough. Now try the following:

0 ! =,   **50 ! =**,   **100 ! =**,   **3 × 4 ! =**, **(3 × 4) ! =**,
± 1 ! =,   **± (1) ! =**,   **(± 1) ! =**.

Can you figure out why your calculator gives these results?

For the second problem, you can reason exactly as we did at the start of the previous problem. There are 10 ways of choosing the president, 9 ways of choosing the secretary (once the president is chosen), and 8 ways of choosing the treasurer (once the president and secretary are chosen). Thus, there are 10×9×8 = 720 ways of choosing the three officers. There are several ways of calculating this with your calculator. One way, of course, is simply to press **10 × 9 × 8 =**. In this case, the calculation is not too hard, but it would be less pleasant if we were counting the number of ways of choosing 10 officers out of a club having 50 members. A shorter way of calculating is to observe that

$$10*9*8 = \frac{10*9*8*7*6*5*4*3*2*1}{7*6*5*4*3*2*1} = \frac{10!}{7!}.$$

Thus, you can calculate the number by pressing **10 ! ÷ 7 ! =**. Similarly, the number of ways of choosing 10 officers out of a club with 50 members is found by pressing **50 ! ÷ 40 ! =**,

giving $3.727604302 \times 10^{16}$. In general, the number of ways of choosing *r* objects out of a set of *n* objects, taking order into account, is called the *number of permutations of n objects taken r at a time*, and is denoted by several notations, one of which is $_nP_r$. (There are other notations, such as P_r^n, $P_{n;r}$, and P(*n,r*).) There is a formula for $_nP_r$, namely $_nP_r = \dfrac{n!}{(n-r)!}$, but the good news is that there is a key on your calculator which gives the result of this calculation in one step. On your calculator, you will find *nPr* in orange above the 6 key. (We will discuss the operation of this key below.) Thus, to find the number of ways of choosing 3 officers in a club with 10 members, you press

> **10 2nd F 6 3 =** ,

giving *720* as before. From now on, we will denote the key combination **2nd F 6** by $_nP_r$. To calculate the number of permutations of 50 objects taken 10 at a time, you press **50 $_nP_r$ 10 =** , giving $3.727604302 \times 10^{16}$ as before.

For the third problem, we observe that the order of the committee members does not matter. Thus, if the committee consists of *A*, *B*, and *C*, it does not matter whether *A* is chosen first, *B* second, and *C* third, or whether *C* is chosen first, *A* second, and *B* third. So to get the number of distinct committees with three members, we have to divide by the number of ways of arranging the three members. This gives $\dfrac{10*9*8}{3*2*1} = \dfrac{10!}{7!*3!} = 120$ ways of choosing the committee. In general, the number of ways of choosing *r* objects out of a set of *n* objects, where order is not taken into account, is called the *number of combinations of n objects taken r at a time*, and is denoted by several notations, one of which is $_nC_r$. (There are other notations, such as C_r^n, $C_{n;r}$, and C(*n,r*), and an old-fashioned notation $\dbinom{n}{r}$, which fortunately is going out of style.)

There is a formula for $_nC_r$, namely $_nC_r = \dfrac{n!}{r!(n-r)!}$, but again there is a key combination on your calculator which will do the job in one step. On your calculator, you will find *nCr* in orange above the 5 key. Thus, to find the number of possible 3-member committees in a club with 10 members, you press

> **10 2nd F 5 3 =** ,

giving *120* as before. We will denote the key combination **2nd F 5** by $_nC_r$.

Many problems involve the functions $_nP_r$ and $_nC_r$ together with other functions. These can, and should, be solved using a single calculation with the calculator.

Example: Find the number of 5-card poker hands drawn from a standard 52-card deck that contain exactly two aces.

Solution: A five-card hand that contains exactly two out of the four aces also contains three out of the 48 non-aces. Clearly the order in which the five cards are dealt does not matter. So the total number of hands is $_4C_2 * {_{48}C_3}$. We can evaluate this on the calculator just as we see it: **4 $_nC_r$ 2 $\times$ 48 $_nC_r$ 3 =** , giving 103776.

We can also solve probability problems this way.

Example: Find the probability that a 13-card hand dealt from a standard 52-card deck contains exactly 5 spades.

Solution: We want to find what proportion of the 13-card hands drawn from a standard deck contain exactly 5 spades (and 8 non-spades). The number of possible 13-card hands drawn from a standard deck is $_{52}C_{13}$. The number of ways of choosing 5 spades from the 13 spades in

the deck is $_{13}C_5$, and the number of ways of choosing 8 non-spades from the 39 non-spades is $_{39}C_8$. So the number of hands containing exactly 5 spades and 8 non-spades is $_{13}C_5 \times {}_{39}C_8$. Thus, the proportion of hands containing exactly 5 spades is

$$\frac{{}_{13}C_5 \times {}_{39}C_8}{{}_{52}C_{13}}.$$

You can punch this in just as you see it:

13 $_nC_r$ 5 × 39 $_nC_r$ 8 ÷ 52 $_nC_r$ 13 = ,

giving **0.124691925**. (You do not need to put parentheses into this calculation, although you may if you wish.)

Since your calculator has keys for $_nP_r$ and $_nC_r$, there is no reason for you to use the formulas involving factorials to evaluate these functions. If you ask, "But if I don't write any factorials or other intermediate calculations on my exam paper, how does the marker know how I got my answer?", my answer again is that you should simply write the key-strokes on your paper. Unless and until examiners specifically start asking you to write your answers in terms of factorials, you might as well use the full power of your calculator. Actually, the important thing examiners are testing for in these kinds of problems is whether you know when to use $_nP_r$ and when to use $_nC_r$. For this, you have to understand what the problem is asking, and your calculator will not help you with that.

(The title of this section is taken from Elizabeth Barrett Browning's Sonnet XLIII, which begins:

"How do I love thee? Let me count the ways."

Her answer does not involve factorials. You may want to read the poem to find out how many ways she finds. You can find it in just about any anthology of English poetry.)

Touching all the Bases

The Sharp (but not the Casio) has one more feature which will be useful to you if you are seriously involved with computers (Computer Science or Computer Engineering). It can represent integers in base 2 (binary), base 8 (octal), base 16 (hexadecimal), and, for reasons that totally escape me, base 5 (pental). Besides the usual operations of addition, subtraction, multiplication, and division, it can perform the Boolean operations NOT, AND, OR, XOR, and NXOR. To put your calculator into base 2, 8, 16, or 5, press **2nd F ÷** (**→BIN**), **2nd F -** (**→OCT**), **2nd F ×** (**→HEX**), or **2nd F =** (**→PEN**) respectively. You will see a small raised ^{b}, ^{o}, ^{H}, or ^{P} at the right of the lower line of your display. To get back to base 10 (decimal), press **2nd F +** (**→DEC**). The raised letters will disappear, and you will see a decimal point at the end of the lower line of the display.

When your calculator is operating in base 2, 8, 16, or 5, many of the keys on your calculator will not operate at all, or will operate differently from the way they operate in decimal mode. In base 2, the number keys 2, 3, 4, 5, 6, 7, 8, 9 will not operate (naturally enough), nor will the black function keys in the second or third rows. In base 8, the same goes for the number keys 8 and 9, and for the function keys. In base 16, all the number keys will operate, and the function keys in the second row will give you the hexadecimal digits A, B, C, D, E, F, which will appear in the display something like A, B, C, D, E, F. In all these bases, the top row of black keys will give the Boolean operations **NOT**, **AND**, **OR**, **XOR**, and **NXOR**. Although these are marked in blue, you do not need to press **ALPHA** to access them. Furthermore, the ± key gives **NEG** (the "2's complement"). By and large, the Boolean operations need no explanation, with

the possible exception of **XNOR**. The effect of **A XNOR B** is the same as **NOT (A XOR B)**. In view of this, perhaps **NXOR** would be a better name for the operation.

In all these bases, the memory operations **RCL**, **STO**, and **ALPHA** applied to memory locations **A**, **B**, **C**, **D**, **E**, **F**, **M**, **X**, **Y** work as usual, as does the **M+** key.

Vital Statistics

In this section, we look at how to use your calculator for statistical calculations. If you are interested in these applications, you probably either are taking a statistics course, or you have taken one and want to use what you have learned. So I assume that you are familiar with the basic concepts and notations of statistics.

Entering Statistics Mode

To enter Statistics Mode, press **MODE**. You will see

NORMAL	**STAT**
0	*1*

Press **1**. You will then see

SD	**LINE**	**QUAD**
0	*1*	*2*

The simplest choice is *0* for Single-variable Data. The remaining options are for studying relationships between two variables, using regression. If you press *1*, your calculator will try to fit your data to a LINEar relation $y = a + bx$. If you press *2*, it will try to fit the data to a QUADratic relation $y = a + bx + cx^2$. If you press the down-arrow key $\Downarrow$, you will see more options:

EXP	**LOG**	**PWR**
3	*4*	*5*

These choices will give an EXPonential relation $y = a * e^{bx}$, a LOGarithmic relation $y = a + b * \ln x$, and a PoWeR relation $y = a * x^b$ respectively. If you press $\Downarrow$ again, you will see

INV
6

for an INVerse relation $y = a + b/x$. Pressing one of the keys **1** - **6** will choose the corresponding kind of regression. In the great majority of cases when you are doing regression, you will choose **1** for linear regression.

Whichever one you choose, you will see

Stat *k*

0.

in the display, where **k** is the value **0** - **6** you chose. You will also see a small ^{STAT} in the upper right-hand corner of the display, to remind you that you are in STATistics mode. While you are in STATistics mode, you will be able to access a number of STATistical VARiables, using the **STAT VAR** (**ALPHA**) key. (We discuss these in detail below.) Certain keys, notably the **STO** and **M+** keys, will have special properties in STATistics mode. On the other hand, you will not be able to access the memory locations **A** - **F**, **X**, **Y**, and **M**.

Entering Data

THE FIRST THING TO DO, WHEN YOU START ENTERING DATA FOR A NEW PROBLEM, IS TO CLEAR OUT ANY DATA THAT MAY ALREADY BE STORED. You do this by pressing **2nd F CA (MODE)**. You can also do it by pressing **2nd F M-CLR** and then

0 0 . You need to do this because if you have entered any data values previously, they remain stored in your calculator, even after you turn it off.

The **M+** key plays a key role in entering statistical data. Below this key, you will see, in gray, the label **DATA**. Henceforth we will refer to this key as the **DATA** key. The simplest situation is when you are in the Single-variable Data sub-mode. Suppose your data values are 79, 93, 84, 86, 77, 63, and 46. You enter them by pressing

79   DATA
93   DATA
84   DATA
86   DATA
77   DATA
63   DATA
46   DATA

DO NOT PRESS THE **=** KEY! After you enter each value, you see

DATA SET=
k.

where *k* is the number of data items you have entered. This helps you keep track of where you are in entering data.

Suppose now you want to enter the data values 81, 81, 81, 81, where the value 81 is repeated four times. Of course, you could do this by entering these items one at a time, as you did above. However, an easier way is to press

81   DATA   DATA   DATA   DATA

But what if a data item occurs 10 times or 100 times? You could, of course, press the **DATA** key as many times as needed, but there is a still easier way. Below the **STO** key, you will see **(*x,y*)** in gray. In STATistics mode, this key is used to separate parts of a data entry, and from here on we will refer to it as the **(*x,y*)** key. Note that it is conveniently located next to the **DATA** key. To enter the data value 73 with a frequency of 10, you press

73   (*x,y*)   10   DATA

Notice that the number ***k*** in the display

DATA SET=
k.

only goes up by 1. The **DATA SET** display counts the number of data sets, not the total of the frequencies, which statisticians refer to as *n*. We will see the significance of this when we examine how to correct data.

If you are only interested in Single-variable Data, you can skip to the section "**Correcting Data**" at this point. If you are in one of the regression sub-modes, each data value consists of a pair of numbers (*x,y*). Go into the LINear regression sub-mode by pressing **MODE 1 1** . (This will clear out any existing data – you do not need to press **2nd F CA** .) Suppose you want to enter the data pairs (79,63), (93,97), (84,87), (63,71). You press

79   (*x,y*)   63   DATA
93   (*x,y*)   97   DATA
84   (*x,y*)   87   DATA
63   (*x,y*)   71   DATA

You will see

DATA SET=
4.

in the display. If you want to repeat the data pair (63,71), you can just press **DATA** as many times as you like. If you want to enter the data pair (70,80) with a frequency of 5, you press

<div align="center">

70   **(x,y)** **80**   **(x,y)** **5**   **DATA**

</div>

Correcting Data

Suppose you are in the Single-variable Data sub-mode, and have entered the values 79, 93, 84, 86, 77, 63, and 46 as above. You may want to check these values over or make corrections. To do this, press the down arrow (⇓). You will see in the display

X1=
<div align="center">79.</div>

Press ⇓ again, and you will see

N1=
<div align="center">1.</div>

The 79 is, of course, your first data value, and the 1 is its frequency. If you do not indicate a frequency before pressing the **DATA** key, your calculator assumes a frequency of 1. Pressing ⇓ repeatedly gives you

X2=
<div align="center">93.</div>

N2=
<div align="center">1.</div>

and so on. When you get to the end of the data values, in this case

N7=
<div align="center">1.</div>

and press ⇓ again, you go back to the beginning with

X1=
<div align="center">79.</div>

Similarly, you can move up the data values by pressing the up arrow (⇑).

Now suppose you want to change the 86 value to a 90. To do this, press ⇓ or ⇑ until you reach

X4=
<div align="center">86.</div>

Now press

90   **DATA**

and you change the X4 value to 90. Similarly, if you move to

N5=
<div align="center">1.</div>

you can change the frequency of the X5 value (which is 77) to 3 by pressing

3   **DATA**

But suppose you want to eliminate a data item, say X3, entirely. To do this, you move to X3 by pressing the ⇓ and ⇑ keys. Then you press

2nd F DATA

(Note the **CD** in orange to the bottom right of the **DATA** key. You can think of this as standing for Cancel Data.) When you do this, the X3 value, with its frequency, is wiped out, and all the other data values, with their frequencies, move up in the list. Thus, you now have X1 = 79, N1 = 1, X2 = 93, N2 = 1, X3 = 90, N3 = 1, X4 = 77, N4 = 3, X5 = 63, N5 = 1, X6 = 46, N6 = 1.

If you are in one of the regression sub-modes, practically everything is the same as what we have just described for Single-variable Data. The only difference is that when you run through the data values using the $\Downarrow$ and $\Uparrow$ keys, you find, for example,

X1=
>
> **79.**

Y1=
>
> **63.**

N1=
>
> **1.**

and so on. You can modify or delete any of these values in the same way as in the Single-variable Data case.

Displaying and Calculating with Statistical Variables

Of course, you go through this process of entering data in order for the calculator to give you various statistical quantities calculated from the data, such as the mean, standard deviation, regression equation, and correlation coefficient. You get these by pressing the **ALPHA** key (note that **STAT VAR** occurs in gray under this key) followed by one of the gray keys in the bottom four rows of the keyboard. Thus, n, the total number of data items, is the sum of the frequencies N1, N2, … . To get it, you press

> **ALPHA**   **0**

Similarly, Σx, the sum of the x values (taking frequencies into account) is the sum of the products X1*N1, X2*N2, … , and is obtained by pressing

> **ALPHA**   **.** (decimal point)

You get Σx^2, the sum of the products X1^2*N1, X2^2*N2, … , by pressing

> **ALPHA**   **+/-**

To get $\bar{x}$ (the mean of the x values), sx (the sample standard deviation), and σx (the population standard deviation) you press

> **ALPHA**   **4**
> **ALPHA**   **5**
> **ALPHA**   **6**

respectively. All of these are available in all of the statistics sub-modes.

In the regression sub-modes, you similarly get Σxy, Σy, Σy^2, $\bar{y}$, sy, σy by pressing

> **ALPHA**   **1**
> **ALPHA**   **2**
> **ALPHA**   **3**
> **ALPHA**   **7**
> **ALPHA**   **8**
> **ALPHA**   **9**

respectively. The coefficients a and b of the regression equation are obtained by pressing

> **ALPHA**   **(**
> **ALPHA**   **)**

respectively. In all the regression sub-modes except QUAD (Stat 2), the correlation coefficient r is obtained by pressing

> **ALPHA**   **÷**

In QUAD, r is not available, but c (the coefficient of x^2 in the regression equation) is obtained by pressing

ALPHA   ×

Given a value of x, to find y', the corresponding value of y predicted by the regression equation, you enter the value of x, and then press

2nd F   **(**

(Note that you use **2nd F**, and not **ALPHA** , for this.) Similarly, given a value of y, to find x', the value of x predicted by the regression equation, you press

2nd F   **)**

In the case of quadratic regression, there are in general two such values of x'. You get one of them by pressing **2nd F**) , as above, and you can flip to the other by pressing

2nd F   **Exp**

(Note the symbols ←,→ in orange above the **Exp** key.)

You can use these statistical quantities in expressions. For example, you can get the sample variance s^2 or the population variance σ^2 in the obvious way. To get s^2, you press

ALPHA   **5**   **x^2**   **=**

and to get σ^2, you press

ALPHA   **6**   **x^2**   **=**

A less obvious example is the following: Suppose you want to calculate the t-statistic

$$t = \frac{\bar{x} - \mu}{\dfrac{s}{\sqrt{n}}}$$

for a test of the hypothesis that the population mean $\mu = 0.5$. You can calculate this by pressing

(ALPHA 4 - 0.5) ÷ (ALPHA 5 ÷ √ ALPHA 0) =

Examples

In this section, we give examples, taken from textbooks, which will enable you to make sure you have mastered the use of your calculator for statistical calculations. We give only the answers.

Problem: Given the following scores on an examination, find n, Σx, Σx^2, $\bar{x}$, s, and σ^2.

79   97   86   76   93   87   98   68   84   88   81   91

Answers: $n = 12$, $\Sigma x = 1028$, $\Sigma x^2 = 88910$, $\bar{x} = 85.7$, $s = 8.76$, $\sigma = 70.4$ (the last three rounded off to three significant figures)

Problem: A pair of dice were rolled 100 times, giving the following values:

Value	2	3	4	5	6	7	8	9	10	11	12
Freq	3	6	9	12	16	18	11	10	7	6	2

Find n, Σx, Σx^2, $\bar{x}$, s, and σ^2.

Answers: $n = 100$, $\Sigma x = 680$, $\Sigma x^2 = 5196$, $\bar{x} = 6.80$, $s = 2.40$, $\sigma^2 = 5.72$

Problem: The following data show scores on a placement test and on an examination for 10 students:

Placement	39	43	21	64	57	47	28	75	34	52
Exam	65	78	52	82	92	89	73	98	56	75

Using linear regression, find n, Σx, Σy, Σx^2, Σxy, Σy^2, $\bar{x}$, $\bar{y}$, s_x, s_y, a, b, r, and the predicted exam score for a student who got 50 on the placement test.

Answers: $n = 10$, $\Sigma x = 460$, $\Sigma y = 760$, $\Sigma x^2 = 23634$, $\Sigma xy = 36854$, $\Sigma y^2 = 59816$, $\bar{x} = 46.0$, $\bar{y} = 76.0$, $s_x = 16.6$, $s_y = 15.1$, $a = 40.8$, $b = 0.766$, $r = 0.840$, $y' = 79.1$

Modes and Setup

Resetting

If your calculator is behaving strangely, and you can't figure out how to get it back on track, the best thing to do is to reset it to the condition it was in when you bought it. To do this, press **2nd F** **ALPHA** (that is, **M-CLR**). When you do this, you see in the display

MEM	**RESET**
0	**1**

Press **1** to reset the calculator to its original state. You will see

RESET?
0

If you press **0** or **=**, you will clear all memories and playback (more about these in the sections "**Thanks for the Memories**" and "**Play it again, Sam**"), and you will also reset the calculator to **NORMAL** mode (as opposed to **STAT** mode), **NORM1** display format, and **DEG** measure for angles. (These settings are all discussed in detail below.) If you change your mind and decide you really don't want to reset the calculator, press anything else (such as **ON/C**).

Similarly, if you do not want to reset the calculator completely, but merely erase all memories and playback, press **0** or **=**   You will see in the display

CLR_MEMORY?
0

If you press **0** or **=**, you will clear all memories and playback. Again, if you change your mind, press anything else.

Normal and Statistics Modes

Your calculator can operate in two modes, **NORMAL** and **STAT** (for STATistics). The **NORMAL** mode is used for everyday calculations. The **STAT** mode is used for statistical analysis. To change modes, press the **MODE** key (in the second row to the right of the arrow keys at the top of the calculator). You will see

NORMAL	**STAT**
0	**1**

Press **0** or **=** to go into (or stay in) **NORMAL** mode. If you press **1** to go into **STAT** mode, you will see a row of further choices:

SD	**LINE**	**QUAD**		**EXP**	**LOG**	**PWR**	**INV**
0	**1**	**2**		**3**	**4**	**5**	**6**

(The last four choices can be reached by using the arrow keys.) If you really want to do statistics, refer to the section "**Vital Statistics**". If that is not what you want, press any one of the keys **0** - **6**, and then press **MODE** to get back into **NORMAL** mode.

Display Formats

Your calculator can handle numbers ranging in absolute value from 1.0×10^{-99} to $9.999999999 \times 10^{99}$ (besides 0, of course). To represent these numbers on the display, it can use a variety of formats. As it comes from the store, or after you have reset it, it uses a format called **NORM1**. In this format, numbers between 0.000000001 (that is, 10^{-9}) and 9999999999 (that is, 10^{10} - 1) in absolute value are represented in decimal form, with 10 significant figures for numbers ≥ 1, and up to 9 significant figures for numbers < 1, with trailing zeroes to the right of the decimal point omitted. Thus, for example, 10/3 is displayed as *3.333333333*, 1/3 as

0.333333333, and 1/8 as **0.125**. Numbers with absolute values $\geq 10^{10}$ or $< 10^{-9}$ are displayed in scientific notation. For example, 6.02×10^{23} is displayed as **6.02 x10²³**, with small characters **x10** to the right of the digits in the display, and the power of 10 in raised characters to the right of that. This format is probably the most useful for most purposes.

However, you may want to have your calculator display numbers in a different format. You can select other formats using the **SET UP** key (in the third row to the left of the arrow keys). If you press this key, you will see (if you are using **NORM1** display format):

FSE   TAB
0

The entry FSE stands for "Fixed, Scientific, Engineering". If you press **0** or **=**, you will see

FIX   SCI   ENG   NORM1   NORM2
0   1   2   3   4

The last two choices can be reached by using the up- or down- arrow keys ⇑ or ⇓ . As we have mentioned, when the calculator is reset to its initial state, numbers are displayed in **NORM1** format. The **NORM2** format is a variation of **NORM1** format in which numbers less in absolute value than 0.01 are displayed in scientific notation. The advantage of **NORM2** format is that more significant figures of the number are displayed than in **NORM1** format. For example, if you enter 1.2345×10^{-8} into your calculator, in **NORM1** format you will see **0.000000012**, whereas in **NORM2** format you will see **1.2345 x10⁻⁰⁸**. You may prefer this format.

In the remaining formats (**FIX**, **SCI**, and **ENG**), you can specify the number of digits to be displayed, using the **TAB** option which appears when you press the **SET UP** key. In **FIX** mode, numbers are displayed (if possible) with a specified number of decimal places. For example, to display numbers in dollars and cents, with two decimal places, press **SET UP** to get the

FSE   TAB
0

display, press **0** to get

FIX   SCI   ENG   NORM1   NORM2
0   1   2   3   4

and press **0** to get **FIX** format. Then, to get two decimal places, press **SET UP** again. This time, you will see

FSE   TAB
0   1

Press **1** to get

TAB(0-9)?

Now press **2** to select two decimal places. If you enter the number 12.345 into the calculator, you will see **12.35**, with two decimal places and with the decimal part 0.345 rounded off to 0.35. However, the calculator still carries all the significant figures, as you can see if you multiply this number by 10, giving **123.45** in the display.

In **SCI** format, the **TAB** option serves to specify the number of digits after the decimal point to be displayed. To get **SCI** format, press **SET UP**, and then 0 to get the **FSE** menu, and 1 to choose **SCI** format. If you now enter 123.45, the display will show **1.23x10⁰²** (assuming you have previously chosen **TAB** = 2). To change this to (for example) three digits to the right of the decimal point, press **SET UP**, then **1** to get the **TAB** choice, and then **3**. The display will now show **1.235x10⁰²**, with the last digit rounded up. The **ENG** format is a variation of the **SCI** format in which the power of 10 is always a multiple of 3. To get **ENG** format, press **SET UP**,

press **0** to get the **FSE** menu, and then **2** to choose **ENG** format. Now the display changes to **123.450$_{×10}$^{00}**. If you multiply this by 10, you will get the display **1.235$_{×10}$^{03}**. Engineers like this notation because it simplifies comparing numbers "in the same ball park". For example, it is not easy at a glance to compare the numbers $2.345×10^5$ and $5.432×10^4$ written in scientific notation. In engineering notation, they become $234.5×10^3$ and $54.32×10^3$, with the same power of 10, and these numbers are easily compared.

Degrees, Radians, Grads

Finally, you have the selection of degrees, radians, or grads. This setting only concerns the trigonometric functions sin, cos, tan, and their inverses. To change from degrees to radians, press the **DRG** key. (This key is the rightmost key on the first row of black keys, and is not to be confused with the **DRG** key, above the π key just to the left of the **DRG** key, nor with the ↔**DEG** key, above the **D°M′S** key, which is the third key in the last row of black keys. The uses of these keys, and much more, are discussed in the section "**Playing the Angles**" above.) When you press the **DRG** key, the small **DEG** at the top of the display changes to **RAD**. When you press it again, the **RAD** changes to **GRAD**, and when you press it a third time, the **GRAD** changes back to **DEG**.

CASIO fx-300MS

Getting Started

In the following, we will indicate keys to be pressed in boldface, thus:

2 + 2 =

and the results which you should see on the bottom line of the display in boldface italic, thus:

4.

You should try each of these examples, to make sure you get the same result.

Let's look at some simple arithmetic calculations. The calculator is turned on by pressing the **ON** key in the upper right-hand corner of the keyboard. Probably you will be able to follow through the rest of this part without any preliminary steps. But –

IF YOU FIND THAT YOUR CALCULATOR IS BEHAVING STRANGELY,
PROCEED TO THE SECTION "**Modes and Setup**" BELOW.

You should see

0.

at the right of the lower line of the display. You should also see, again in small letters at the very top, "**D**" or "**R**" or "**G**", with a flashing cursor at the left of the first full-size line of the display.

IF YOU SEE ANYTHING ELSE, PROCEED TO "**Modes and Setup**".

Now press the following keys:

3 + 4 × 5 =

You should see

3 + 4 × 5

in the first line of the display, and

23.

at the right of the second line. Notice that the calculators use normal conventions for evaluating mathematical expressions, with multiplications (and divisions) evaluated before additions and subtractions. If you are using some other type of calculator, and you see

35.

at the right of the second line, you are probably using a so-called "business calculator" (or perhaps a very old model). Get rid of it, and get yourself a scientific calculator.

Now press the following keys:

- 6 ÷ 7 =

You should see

Ans - 6 ÷ 7

on the first line, and

22.14285714

on the second line. What has happened is that the calculator assumes you want to take the previous result (namely, 23) as the starting point for the second calculation, and subtracted 6/7 from it. This will always happen when you start a new calculation with one of the keys +, -, ×, or ÷. It will also happen under certain other circumstances, which we will discuss below.

If you do not want this to happen, and you want to calculate -6/7 by itself, there are two methods. One is to clear out the previous calculation by pressing **AC** and then pressing

- 6 ÷ 7 =

giving

-0.857142857

as a result.

A better method is the following: Press

(-) 6 ÷ 7 =

giving the same result. (The key **(-)** is at the left of the third row of black keys.) Notice the difference between the subtraction key **-** on the one hand, and the "change-sign" key **(-)** on the other. The subtraction key **-** is used between two expressions to indicate that the second is to be subtracted from the first. The "change-sign" key **(-)** precedes an expression to indicate that its sign is to be changed (from negative to positive or from positive to negative). Conceptually, these are two different operations, and you should get in the habit of using the one you really want. Sometimes you can get away with using one in place of the other, but sometimes you can't.

We saw that the calculator interprets 3 + 4×5 (correctly) as 3 + (4×5). But what if you want to calculate (3 + 4)×5? One way is to press **3 + 4 =** to give **7.** and then **× 5 =** to give **35.** as a result. As we have seen, your calculator will use the answer from the first calculation as the starting point for the second. But what if you want $3 \times (4 + 5)$ or $\dfrac{3+4}{5+6}$? This is where the parenthesis keys **(** and **)** come in handy. These keys are located above the **9**, in the middle of the bottom row of black keys. To calculate $(3 + 4) \times 5$, you press

(3 + 4) × 5 =

giving

$$35.$$

To calculate $3 \times (4 + 5)$, press

3 × (4 + 5) =

to give **27.** as the result. To calculate $\dfrac{3+4}{5+6}$, you have to realize that the fraction bar means that the whole numerator, namely 3 + 4, is to be divided by the whole denominator, namely 5 + 6. Thus, what you want is (3 + 4)/(5 + 6), so you punch it into the calculator accordingly:

(3 + 4) ÷ (5 + 6) =

giving **0.636363636** as the result. The same idea works for more complex calculations. To calculate

$$\frac{\dfrac{3+4}{5+6}+\dfrac{9+8}{7+6}}{\dfrac{3+5}{4+6}-\dfrac{9+7}{8+6}},$$

you should think of the expression as

((3 + 4)/(5 + 6) + (9 + 8)/(7 + 6))/((3 + 5)/(4 + 6) - (9 + 7)/(8 + 6)),

and punch it in accordingly:

((3 + 4) ÷ (5 + 6) + (9 + 8) ÷ (7 + 6)) ÷ ((3 + 5) ÷ (4 + 6) - (9 + 7) ÷ (8 + 6)) =,

giving **-5.67016317** (try it!). In this case, it would be easier to perform the additions 3 + 4, 5 + 6, …, mentally, and then do the divisions, additions, and subtractions with the calculator, but if the additions were, say, 3.4567 + 4.321, 5.987 + 6.123, …, you probably would want the calculator to do the work. At any rate, the calculator can handle more levels of parentheses than you are likely to toss at it.

It is not too hard to remember to put parentheses around the denominator when the denominator is a sum or a difference, as in the above example. But it is easy to forget parentheses when the denominator is a product or a quotient. Consider the following example:

$$\frac{3+4}{5\times 6}.$$

It seems reasonable to punch this in as

(3 + 4) ÷ 5 × 6 =   NO!!!

giving **14.4** instead of 7/30 = 0.2333333333… . Why does this happen? To see the reason, consider the expression 7 - 5 + 6. You interpret this automatically as (7 - 5) + 6 = 8, not as 7 - (5 + 6) = -4. That is, in a sequence of mixed additions and subtractions, you perform the operations in sequence from left to right. You do not encounter sequences of mixed multiplications and divisions such as $7 \div 5 \times 6$ as frequently, but your calculator treats them in the same way as it treats mixed additions and subtractions. Thus, it interprets **(3 + 4) ÷ 5 × 6** as **((3 + 4) ÷ 5) × 6**, and arrives at the result **14.4**, as we have seen. But that is not what you want, so you have to punch in

(3 + 4) ÷ (5 × 6) =   YES!!!

to get the desired answer **0.233333333** .

One more thing. In ordinary algebra, when you put two expressions next to each other without any operation symbol between them, you understand that a multiplication is to be performed. Thus, when you write 3(4 + 5), you understand that this means $3 \times (4 + 5)$. The calculator will also understand this – sometimes. Thus, if you punch in

3 (4 + 5) =

you will get

<div align="center">

27.

</div>

as you would expect. But if you punch in

(3 + 4) 5 =

you get

Syntax ERROR

as your result. Go figure! At any rate, to get rid of the error message, press AC . The best thing, once again, is to punch in what you really want, which is

(3 + 4) × 5 = .

Editing Expressions

But what if you make a mistake punching in your expression? One way out is to press AC and re-do the expression from the beginning. But if your expression is long, like the fraction example above, you may not want to do this. Fortunately, there is another way. Suppose you have punched in

3 + 4 × 5 =

and you realize that you really want to have

3 - 4 × 5 = .

This is where the arrow keys come in. They are located at the top of the keyboard, just below the display. We will denote them by $\Leftarrow$, $\Uparrow$, $\Rightarrow$, and $\Downarrow$. For the moment, we consider only the left arrow ($\Leftarrow$) and right arrow ($\Rightarrow$). (We discuss the up arrow ($\Uparrow$) and down arrow ($\Downarrow$) in the sections "**Play it again, Sam**" and "**Vital Statistics**" below.)

If you punch in the keys **3 + 4 × 5 =** , you will see on the top line

3 + 4 × 5

and at the right of the bottom line

23.

If you press ⇐ (the left arrow key), a flashing underline (_) will appear at the end of the top line. At this point, you can add more symbols to the expression. If, for example, you press **+ 6 × 7 =** , the top line becomes

3 + 4 × 5 + 6 × 7

and on the bottom line you will have the new result

65.

Now suppose you want to replace the first **+** by a **-** , so that your expression will be **3 - 4 × 5 + 6 × 7** . Press ⇒ (the right arrow key). This time, the flashing cursor will appear at the left end of the line, over the **3** . Press ⇒ again. Now, the flashing cursor is on the **+** sign between the **3** and the **4** . Press **-** . The **+** will be changed to **-** , and the flashing underline cursor will move to the **4** . If you now press **=** , you will see on the top line

3 - 4 × 5 + 6 × 7

and on the bottom line

25.

In this way, you can replace numbers or symbols with other numbers or symbols. The **DEL** key (the red key to the left of the **AC** key, in the fourth row from the bottom) will delete the key under the cursor.

But what if you want to insert a number or symbol into the expression? To do this, press **SHIFT** (at the top left of the keyboard), and then the **DEL** key. (Note that INS is written, in orange, above the **DEL** key.) The cursor changes to a hollow rectangle **~** , and you can insert numbers or symbols. In this mode, the **DEL** key acts as a backspace, erasing the character before the cursor. If you press **=** , to see your new result, and then press ⇒ or ⇐ to make more changes, you will go back to the flashing underline, and the keys you press will write over the existing keys.

You can flip back and forth between the underline cursor (which writes over existing symbols) and the hollow rectangular cursor (which inserts symbols into existing expressions) by pressing **SHIFT DEL** (**INS**). The calculator will stay in whichever mode you choose, even if you turn the calculator off and on, until you press **INS** again.

There are two things to watch out for. First, when the flashing cursor is the underline (write-over mode), the **DEL** key erases the symbol under the cursor, but when the flashing cursor is the rectangle (insert mode), the **DEL** key works like a backspace, erasing the symbol to the left of the cursor. Second, to start making additions or changes to the *right* end of the expression, press ⇐ (the *left* arrow key). To start making changes to the *left* end of the expression, press ⇒ (the *right* arrow key). The idea seems to be that you press ⇐ if you want to make corrections from right to *left*, and ⇒ if you want to make corrections from left to *right*.

To make more changes, move the flashing cursor to the right or left, using the arrow keys, and make whatever changes you like. Note that the length of the expression is not limited by the twelve characters that can be displayed at any one time. The display will "slide along" so that the cursor is always in view, to a maximum of 79 characters. This should be enough for any reasonable calculation.

Experiment with making insertions and deletions until you feel comfortable with this aspect of your calculator's operation.

Squares, Cubes, Reciprocals, Square Roots, Cube Roots

Your calculator can calculate squares (x^2), cubes (x^3), reciprocals ($\frac{1}{x}$), square roots

($\sqrt{x}$), cube roots ($\sqrt[3]{x}$), as well as arbitrary powers (y^x) and roots ($\sqrt[x]{y}$). You will find keys
marked **x^1** and **x^3** as the first and last keys in the top row of black keys, and keys marked **√x** and
x^2 as the second and third keys in the second row of black keys. To get cube roots, press **SHIFT**
and then **x^3** (which has **$\sqrt[3]{x}$** above it).

The operation of these keys is straightforward. To get squares, cubes, and reciprocals,
you press the corresponding key *after* entering the number to be squared, cubed, or inverted.
Thus you have, for squares,
>   **5 x^2 =**
>>   **25.**

for cubes,
>   **5 x^3 =**
>>   **125.**

and for reciprocals,
>   **5 x^{-1} =**
>>   **0.2**

To get square roots and cube roots, you press the corresponding key *before* entering the
number. Thus we have, for square roots,
>   **√ 5 =**
>>   **2.235067077**

and for cube roots,
>   **SHIFT x^3 =**
>>   **1.709975947**

Of course, it is never necessary to use the reciprocal function key to find the reciprocal of
a number. You can always find $1/x$ in the obvious way, by pressing **1 ÷** . However, it is
sometimes convenient to use the reciprocal function. For example, to calculate
1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7 + 1/8, you can press
>   **1 + 2 $^{-1}$ + 3 $^{-1}$ + 4 $^{-1}$ + 5 $^{-1}$ + 6 $^{-1}$ + 7 $^{-1}$ + 8 $^{-1}$ =**

to get **2.717857143** .

If you do not specify the number whose square, cube, reciprocal, square root, or cube root
you want, the calculator will assume that you want to use the answer from the previous
calculation, if there is one. Thus, if you have calculated **3 + 4 × 5 =** to get a result of **23.** , and
you press **x^2** , you will see **ANS^2** in the first line of the display. If you then press **=** , you will see
529. as your answer; similarly if you press **x^3** or **x^1** . Likewise, again supposing that your
previous answer was **23.** , if you press **√ =** , you will see **√ANS** on the first line, and
4.795831523 on the second line; similarly if you press **$\sqrt[3]{}$ =** .

Example: Solving Quadratic Equations

As you know, the roots of a quadratic equation $ax^2 + bx + c = 0$ are given by
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

You can calculate the roots using this formula, remembering that the numerator, the denominator, and also the discriminant b^2 - $4ac$ must be enclosed in parentheses for calculation.

Problem: Find the roots of the equation $3x^2 + 4x - 5 = 0$.

Solution: You can calculate one of the roots as follows:

((-) 4 + √ (4 x^2 - 4 × 3 × (-) 5)) ÷ (2 × 3) =

giving **0.786299647** . You can easily calculate the other root by pressing ⇒ , then pressing ⇒ repeatedly until the cursor is on the **+** sign, pressing **-** to replace **+** by **-** , then pressing **=** to get **-2.119632981** .

If the equation has no real roots, because the discriminant b^2 - $4ac$ is negative, you will get the error message **Math ERROR**. In this case, you can press the left or right arrow keys to bring you to the point in the expression where the error lies (or at least where the calculator thinks the error lies – sometimes the cause of the error lies elsewhere).

Other Powers and Roots

To calculate other powers, you use the **^** key. For example, to calculate 3^5, you press

3 ^ 5 =

to get **243** . Both the base and the exponent can be decimal fractions. For example, if you press

3 . 4 5 ^ 6 . 7 8 = ,

you get **4430.099337** .

The base can be a negative number, provided that the exponent is an integer (whole number, positive or negative):

((-) 3 . 4 5) ^ 6 =

gives **1686.221298** . Note the parentheses! If you press

(-) 3 . 4 5 ^ 6

your calculator will interpret the expression as $-(3.45^6)$, giving **-1686.221298**. Once again, the remedy is to punch in what you want, and not leave it up to the calculator to decide what it thinks you want.

The base can also be negative when the exponent is a fraction whose denominator is odd. For example, you can have

(-) 5 ^ (1 ÷ 3) =

giving **-1.709975947** . You can even replace **(1 ÷ 3)** by **3** **$^{-1}$** . Try it! But

((-) 5) ^ . 3 3 3 =

gives **Math Error** , because the calculator interprets .333 as 333/1000. Likewise, the exponent can be negative:

3 . 4 5 ^ (-) 6 . 7 8 =

gives **2.257285716** $_{×10}$ $^{-04}$. If you like, you can always calculate squares, cubes, reciprocals, square roots, and cube roots using the exponents 2 , 3 , -1 , 1/2 , and 1/3 . But it is quicker and easier to use the special keys provided for these frequently-occurring powers.

The procedure for calculating roots is much the same. To calculate $\sqrt[5]{3}$ (the fifth root of 3), you press

5 SHIFT ^ 3 =

to get **1.24573094** . This works exactly the same way as $3^{(1/5)}$, and the same remarks concerning negative numbers hold.

You can also use the answer from the previous calculation in these functions. Suppose again that **23.** is the result of the previous calculation. If you press **^** , you will see **Ans^** in the first line of the display. If you now press (for example) **4 =** , you get **279841.** as our answer,

since $23^4 = 279841$. But what if you want to use the previous answer as our exponent, say to calculate 2^{23}? You start out by pressing **2 ^** in the usual fashion. You can then call up the previous answer to serve as the exponent. You do this by pressing the key marked **Ans** (to the left of the **=** key). You then press the **=** key. Thus, the key sequence is as follows:

        2 ^ Ans =

giving **2^Ans** in the top line, and **8388608.** in the bottom line as the new answer.

      Similarly, you can use the previous answer in a root calculation. Suppose again that **23.** is the result of the previous calculation. It is unlikely that you would want to calculate the 23rd root of some expression, but if you did, you could just press **SHIFT ^** to put **ANS** in the expression. More likely, you would want to calculate some root of the previous answer. Say you wanted the fifth root of your previous answer. You would then press

        5 SHIFT ^ Ans =

to get **1.872171231** as the new answer.

Example: Compound Interest

      The formula for compound interest is $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$, where A is the final Amount, P is the starting Principal, r is the annual Rate of interest, n is the Number of times interest is compounded per year, and t is the Time in years.

      Problem: What will be the value at maturity of an investment of $10000 at 7% interest for 5 years, if interest is compounded (a) yearly (b) quarterly (c) monthly (d) daily (e) hourly?

      Solution: Here you have $P = \$10000$, $r = 0.07$, and $t = 5$. In parts (a), (b), (c), (d) you have $n = 1$, $n = 4$, $n = 12$, $n = 365$, and $n = 24 \times 365$ respectively. You punch the numbers into the calculator just as you see them. In part (a), you don't need to punch in the 1 for n, but you might as well do it anyway, because it will be useful for the later parts:

      10000 × (1 + 0.07 ÷ 1) ^ (1 × 5) =

giving **14025.51731**. Thus, the value at maturity is $14025.52 (rounding off to the nearest cent). For part (b), all you have to do is to edit the expression, replacing $n = 1$ by $n = 4$, giving

      10000 × (1 + 0.07 ÷ 4) ^ (4 × 5) =

giving **14147.78196**, or $14147.78. Replacing 4 by 12, you get **14176.2526**, or $14176.25; replacing 12 by 365, you get **14190.19929**, or $14190.20. For part (e), you replace 365 by (24×365) (don't forget the parentheses!), so that you have

      10000 × (1 + 0.07 ÷ (24 × 365)) ^ ((24 × 365) × 5) =

giving **14190.65559**, or $14190.66. Notice that you only need to change the values of n, and you do not need to calculate 24×365 separately. You can, and should, leave that to the calculator. If you feel like it, you can try calculating the amount if interest is compounded "minutely", that is, every minute.

      In general, you should try to do your calculations using only a single expression, without writing down any intermediate results. The following example (taken from real life) shows what can happen when you don't take full advantage of what the calculator can do. Suppose you have the previous example, with monthly compounding. You punch in

      1 + 0.07 ÷ 12 =

and get **1.005833333**. You then write down (incorrectly) 1.0583 on your paper, and punch in

      1.0583 ^ 60 =

to get **29.95903136**, and finally

× 10000 =

to get **299590.3136**, or $299590.31. Your $10000 has grown to almost $300000 in just 5 years! How did you do it? Your first mistake was to miscopy your result from the first calculation. But that is not all. You might have copied it down correctly onto your paper, but punched it back into the calculator wrong. If you had avoided these mistakes, but still used the rounded-off value 1.00583, punching in

1.00583 ^ 60 =

you would have gotten **1.417343406**; multiplying by 10000 would give **14173.43406**, or $14173.43. This is pretty close to the right answer, but it is still not right. If you had not rounded off you answer, and used 1.005833333, you would have gotten the same answer as above, but at the cost of additional effort, and running the risk of making a slip, either in writing the number down or in punching it back into the calculator. The point is that you paid for the calculator (at least I hope you did!), and you might as well take full advantage of its capacities. The simplest and safest thing is to do the calculation in a single step.

You may ask "If I don't write any intermediate values on my paper during an exam, how does the marker know how I got the answer?" My answer is that you can write the key-strokes on your paper, just as you have been doing in these examples. If that doesn't satisfy the marker, nothing will.

Notice that as n increases, the final amount A increases, but it increases more and more slowly, so that it seems to be approaching a fixed value. We will discuss this in more detail when we discuss "continuous compounding" in the section "***Exponentials and Logarithms***" below.

Here is an example showing how to solve for P, when you are given A:

Problem: How much do you need to invest now in order to have $5000 three years from now, with interest at 4.5% compounded monthly?

Solution: You are given A = $5000, r = 0.045, n = 12, and t = 3; you want to solve for P.

There are two approaches. One is to calculate $\left(1+\dfrac{r}{n}\right)^{nt}$, and then divide the result into A. Doing it this way, you punch in

(1 + 0.045 ÷ 12) ^ (12 × 3) =

giving **1.144247832** . Then you punch in

5000 ÷ Ans =

giving **4369.682738**, or $4369.68.

The other way is to do the whole thing in one calculation, punching in

5000 ÷ (1 + 0.045 ÷ 12) ^ (12 × 3) =

giving the same answer. The advantage to the second method is that if you want to re-do the calculation using, for example, r = 5%, you only need to change 0.045 to 0.05 without having to punch in the rest of the expression.

In the section "***Exponentials and Logarithms***" below you will see how to solve for r and for t when the other variables are given.

Example: Future Value of a Series of Payments

The formula for the future value of a series of payments is

$$FV = PMT\,\frac{\left(1+\dfrac{r}{n}\right)^{nt}-1}{\dfrac{r}{n}},$$

where *FV* is the Future Value of the series of payments (with accumulated interest), *PMT* is the amount of each periodic PayMenT, and *r*, *n*, *t* are as before, namely, the annual interest Rate, the Number of times interest is compounded per year, and the Time in years.

      Problem: Suppose you deposit $100 per month into an account paying 4.5% compounded monthly. How much will you have in the account after 5 years?

      Solution: Here you have *PMT* = $100, *r* = 0.045, *n* = 12, and *t* = 5. You punch the numbers into the calculator just as you see them in the formula, making sure to insert parentheses properly:

      100 × ((1 + 0.045 ÷ 12) ^ (5 × 12) - 1) ÷ (0.045 ÷ 12) =

giving ***6714.555214*** , for an answer of $6714.56. It is always a good idea to check to see whether your answer is reasonable. In this case, you deposited $100 per month for 60 months, so you should have something over $6000. The extra $714.56 is the interest you have received during the five years. This looks reasonable. If the calculator had given $671.46 or $67145.55, for example, you would conclude that the answer is unreasonable, and you would check the calculation to see what had been punched in wrong.

Example: Monthly payment on a Loan

      The formula for the present value of a series of payments to be made in the future is

$$PV = PMT\,\frac{1-\left(1+\dfrac{r}{n}\right)^{-nt}}{\dfrac{r}{n}},$$

where *PMT*, *r*, *n*, *t* are as above, and *PV* is the Present Value of the payments.

      Problem: You borrow $5000 at 9% interest for three years to buy a car. What are your monthly payments?

      Solution: Here you have *PV* = $5000, *r* = 0.09, *n* = 12, and *t* = 3; you want to solve for *PMT*. As in the second compound interest example above, you have two choices. You can either calculate the messy algebraic expression on the right of the formula, and then divide it into 5000, using the ANS function, or you can calculate the whole thing at once, dividing 5000 by the expression on the right. Using the first method, you punch in

      (1 - (1 + 0.09 ÷ 12) ^ ((-) 12 × 3)) ÷ (0.09 ÷ 12) =

giving ***31.44680525*** . You then press

      5000 ÷ Ans =

to get ***158.9986633*** , or $159.00 as your monthly payment. Using the second method, you punch in

      5000 ÷ ((1 - (1 + 0.09 ÷ 12) ^ ((-) 12 × 3)) ÷ (0.09 ÷ 12)) =

giving the same result.

      Let's check to see whether this is reasonable: For the 36 months, according to this, you pay a total of $159.00 × 36 = $5724.00. This covers your $5000 loan, with $724 interest. This

seems about right. If you had calculated a monthly payment of $15.90 or $1590.00, you would see that there was something wrong.

Exponentials and Logarithms

Keys for Functions and Their Inverses

Your calculator has two keys for finding exponentials and logarithms. One of them, marked **log** , gives logarithms to the base 10; above it, in orange, you find 10^x . The other, marked **ln** , gives logarithms to the base $e = 2.718\ldots$; above it, in orange, you find e^x . Just as in the case of the function $^3\sqrt{}$, the functions 10^x and e^x are accessed by pressing **SHIFT** **log** , and just as in the case of the functions x^3 and $^3\sqrt{}$, the functions on the same key "undo each other", or, in technical language, are *inverses* of each other. Thus, if you press, say, **3.45 =** and then x^3 **=** , you get **41.063635** . If you then press **SHIFT** x^3 **=** , you get **3.45** back as a result. Likewise, if you press **3.45 =** and then **SHIFT** x^3 **=**, you get **1.511029808** ; if you then press x^3 **=** , you get **3.45** back. Similarly, if you press **3.45 =** and then **log =** , you get **0.537819095** ; if you then press **SHIFT** **log =** , you get **3.45** back. The same thing happens for the **ln** key. You also see the same relationship on the **sin** , **cos** , and **tan** keys, where the **SHIFT** keys give the inverse trig functions. The only exception to this pattern is with the functions x^2 and $\sqrt{}$, which are inverses of each other, but occupy separate keys. The reason undoubtedly is that both x^2 and $\sqrt{}$ are used so often that it saves time for the user not to have to press **SHIFT** for one of them. (On some calculators, the key corresponding to **SHIFT** is marked **INV** , for Inverse.) From now on, we will write $^3\sqrt{}$, 10^x , e^x instead of writing out **SHIFT** each time.

Entering Numbers in Scientific Notation

This is the place to mention the procedure for punching in numbers in scientific notation. One way to enter, for example, 6.02×10^{23} is to press

6 . 0 2 ⨯ 1 0 ^ 2 3 = .

A simpler way is to use the 10^x function:

6 . 0 2 ⨯ 10^x 2 3 = .

But the simplest and best way is to use the **EXP** key, in the middle of the bottom row of gray keys:

6 . 0 2 EXP 2 3 = .

When you do this, you will see in the display

6.02E23=

$$6.02x10^{23} \, .$$

If you want to enter an exact power of 10, say 10^{-33}, it is even simpler:

EXP (-) 3 3 = .

You see **1.**$x10^{-33}$ in the display.

Why *e*? Why Natural Logarithms?

In the old days (before 1980), we used tables of logarithms to the base 10. The way it worked was this: To find, say, 23.45×345.6, we wrote 23.45 and 345.6 in scientific notation as 2.345×10^1 and 3.456×10^2. Our tables of logarithms had logarithms of numbers between 1.000 and 9.999; the values of the logarithms were between 0.0000 and 0.9999. So we looked up the

logarithms of 2.345 and 3.456, finding 0.3701 and 0.5386. The logarithms of 23.45 and 345.6, taking the powers of 10 into account, were 1.3701 and 2.5386. We then added these logarithms, getting 3.9087. We looked in our table to find the number whose logarithm was 0.9087, namely 8.104. Thus, our answer was 8.104×10^3, or 8104. (to four significant figures). (There were other complications, involving what were called "proportional parts", but I won't bore you with them.) The whole process was extremely unpleasant (ask your parents and grandparents!), and the results were only good to four significant figures. Fortunately, all this is obsolete. Logarithms are still important, partly because they are now so easy to calculate, but logarithms to the base 10 are rarely if ever used. Instead, we use logarithms to the base $e = 2.718…$. In fact, if you see "log" in any advanced textbook, say beyond the first-year university level, you should assume that logarithms to the base e are meant, unless there is some statement to the contrary. These logarithms are called "natural logarithms", and often denoted by "ln", but it is not obvious at first glance what is "natural" about them.

     It turns out the number e comes up in many different situations, any one of which could serve as its definition. Here is one which is convenient for us: If you calculate the value of the expression $(1 + x)^{1/x}$ for values of x close to 0, you can get a table like the following:

x	$(1 + x)^{1/x}$
-0.1	2.867971991
-0.01	2.731999026
-0.001	2.719642216
-0.0001	2.718417755
-0.00001	2.718295420
-0.000001	2.718283188
…	…
0.000001	2.718280469
0.00001	2.718168237
0.0001	2.718145927
0.001	2.716923932
0.01	2.704813829
0.1	2.593742460

You see that as x gets close to 0, the values of $(1 + x)^{1/x}$ seem to be getting close to a value which is approximately $2.71828…$. It can be shown mathematically that this is the case, and that the number, to a better approximation, is $2.718281828459…$. The German-Swiss mathematician Leonhard Euler (pronounced "Oiler") was the first to recognize the importance of this number and give it a name. He chose to call it e, undoubtedly thinking that it would be immodest for him to give it the name E, as if he were naming it after himself.

     A good example of the use of this number as a base of exponentials is given by compound interest. You recall that the formula for compound interest is $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$. Using

the laws of exponentials, you can write this formula as $A = P\left(\left(1 + \dfrac{r}{n}\right)^{\frac{n}{r}}\right)^{rt}$. As n gets large, that

is, as you compound more and more frequently, the value of $\dfrac{r}{n}$ gets close to 0, and the value of

$\left(1+\dfrac{r}{n}\right)^{\frac{n}{r}}$ gets close to e, since $\dfrac{n}{r}=\dfrac{1}{\frac{r}{n}}$. The formula $A=Pe^{rt}$ is gives what is called

"continuously compounded interest", that is, the limiting value towards which the compound interest tends as you compound interest more and more frequently.

Example: Continuously Compounded Interest

Problem: What is the value at maturity of a loan of $10000 at 7% interest compounded continuously for 5 years?

Solution: We saw above that the final amounts for interest compounded yearly, quarterly, monthly, daily, and hourly were respectively $14025.52, $14147.78, $14176.25, $14190.20, and $14190.66 respectively. Using the formula for continuously compounding interest, we have $A = 10000 \times e^{(0.07 \times 5)}$. You can punch this into the calculator just as you see it:

10000 × e^x (0.07 × 5) = ,

giving 14190.67549 , or $14190.68. Thus, the value at maturity for continuously compounding interest is only $0.02 more than for hourly compounding, and only $0.48 more than for daily compounding.

You can also solve for r and t, using natural logarithms.

Problem: How long will it take for a principal of $5000 to increase to $8000 at 5.5% interest compounded continuously?

Solution: You have $A = \$8000$, $P = \$5000$, and $r = 0.055$. So the formula for continuously compounded interest gives $8000 = 5000 \times e^{0.055t}$, and you want to solve for t.

Dividing by 5000, you get $\dfrac{8000}{5000}=e^{0.055t}$. Now, as we have said, $\ln x$ is the inverse function of e^x, so you can get rid of the exponential with base e by taking the natural logarithm of both sides:

$\ln\left(\dfrac{8000}{5000}\right)=0.055t$. Now, to solve for t, all that remains is to divide both sides by 0.055, giving

$t=\dfrac{\ln\left(\dfrac{8000}{5000}\right)}{0.055}$. You can punch this into the calculator just as you see it:

ln (8000 ÷ 5000) ÷ 0.055 = ,

giving **8.545520532** , or about 8.55 years.

Example: Solving for t

Now let's see what happens when you use interest compounded monthly instead of continuously:

Problem: How long will it take for a principal of $5000 to increase to $8000 at 5.5% interest compounded monthly?

Solution: This time, you have, besides the above, $n = 12$. So the ordinary compound interest formula gives $8000 = 5000\left(1+\dfrac{0.055}{12}\right)^{12t}$, and again you want to solve for t. As before,

you start by dividing both sides by 5000: $\frac{8000}{5000} = \left(1+\frac{0.055}{12}\right)^{12t}$. To get rid of the exponential,

you take the natural logarithm of both sides. Using one of the properties of logarithms, you get

$\ln\frac{8000}{5000} = 12t\ln\left(1+\frac{0.055}{12}\right)$. To solve for t, you divide both sides by $12\ln\left(1+\frac{.055}{12}\right)$ to get

$$t = \frac{\ln\frac{8000}{5000}}{12\ln\left(1+\dfrac{0.055}{12}\right)}.$$

Again, you can punch this into the calculator as you see it, but this time you have to remember to enclose the denominator in parentheses:

ln (8000 ÷ 5000) ÷ (12 × ln (1 + 0.055 ÷ 12)) = ,

giving **8.56589097** , or about 8.57 years. The answer isn't all that much different from the answer using continuous compounding, but the calculation using continuous compounding is much simpler. One more comment: In this problem, you could use logarithms to the base 10, replacing the **ln** key by **log** (that is, $\log_{10}$), and you would get exactly the same answer. (Try it!) But in the preceding problem you need to use the **ln** key. In general, any problem that can be done using logarithms and exponentials to the base 10 can also be done, in the same way, using logarithms and exponentials to the base e, but some problems can be done better and more easily using logarithms and exponentials to the base e. I do not know of any problems in science, engineering, or business which are better done using base 10. So I suggest that you get in the habit of using only the **ln** and e^x keys, instead of the **log** and 10^x keys.

Example: Finding t using the Present Value Formula

The following problem involves a complicated calculation, but the result is worth seeing.

Problem: (from real life!) Recently I had a credit card balance of $3130.90. My credit card company told me that I could make a minimum payment of $62.00, with an annual interest rate of 18.5% (that's right!) compounded monthly. How long would it take for me to pay off this balance with a monthly payment of $62.00, assuming I made no more charges to my credit card? How much interest would I end up paying?

Solution: The present value formula with $PV = \$3130.90$, $PMT = \$62.00$, $r = 0.185$, and $n = 12$ gives

$$3130.90 = 62.00\frac{1-\left(1+\dfrac{0.185}{12}\right)^{-12t}}{\dfrac{0.185}{12}}.$$

You want to solve for t. Dividing by 62.00 and multiplying by 0.185/12, you get

$$\frac{3130.90\times 0.185}{62.00\times 12} = 1-\left(1+\frac{0.185}{12}\right)^{-12t}$$

Transposing terms, you get

$$\left(1+\frac{0.185}{12}\right)^{-12t} = 1-\frac{3130.90\times 0.185}{62.00\times 12}.$$

Taking logarithms of both sides, you get

$$-12t\ln\left(1+\frac{0.185}{12}\right)=\ln\left(1-\frac{3130.90\times0.185}{62.00\times12}\right).$$

To solve for t, divide both sides by $-12\ln\left(1+\dfrac{0.185}{12}\right)$ to get:

$$t=-\frac{\ln\left(1-\dfrac{3130.90\times0.185}{62.00\times12}\right)}{12\ln\left(1+\dfrac{0.185}{12}\right)}.$$

You can punch this in as you see it, with parentheses around the denominator:

(-) ln (1 - (3130.90 × 0.185) ÷ (62 × 12)) ÷ (12 × ln (1 + 0.185 ÷ 12))

giving **8.210802828**, or almost 8 years and 3 months (8.25 years). In 8 years and 3 months, or 99 months, at $62.00 per month, I would pay $99 \times$ $62.00 = $6138.00 in all, including $3007.10 interest.

I paid off my balance in full, and took a trip with the money I saved.

Thanks for the Memories

Memory Locations

Your calculator has memory locations in which values can be stored, recalled for display, or used in expressions. These memory locations are named **A**, **B**, **C**, **D**, **E**, **F**, **M**, **X**, and **Y**. You will find **A** through **F** above and to the right of each key in the second row of black keys; you will find **X**, **Y**, and **M** to the upper right of the fourth, fifth, and sixth of the third row of black keys. You use these memory locations to save yourself the trouble of punching in the same data value more than once, copying down intermediate results on paper, and then re-entering them into an expression. Along with this, you avoid the risk of punching in, copying down, or re-entering these values incorrectly.

To store a value appearing in the display into a memory location, you press **SHIFT RCL**, that is, **STO**, followed by the key with the name of the memory location. To recall it into the display, you press **RCL** followed by the memory location. To use the value contained in a memory location in an expression, you press **ALPHA** followed by the memory location.

To clear a memory location, you simply store the value 0 into it. For example, to clear memory location **C**, you press **0 STO C**.

The values stored in the memory locations remain unchanged when you press **AC**, or even when you turn the calculator off. This is fortunate, because the calculator automatically turns itself off after five minutes or so, to save the battery, if you have not pressed a key. All the memory locations, including **M**, are cleared when you press **SHIFT MODE** (that is, **CLR**), and then **1 =**. You can find more information about clearing your calculator in the section "**Modes and Setup**" below.

Example: Solving Quadratic Equations using Memory Locations

In the section "**Powers and Roots**" above, you used the calculator to find the solutions of quadratic equations. When the coefficients of the quadratic equation have many significant digits, you can make things easier by using the memory locations.

Example: Find the roots of the quadratic equation $ax^2 + bx + c = 0$ when $a = 2.345$, $b = 6.543$, and $c = -9.876$.

Solution: In the formula for the roots of a quadratic equation, each of the coefficients a, b, and c occurs twice. Rather than punch each one in twice, save them in memory locations **A**, **B**, and **C**. Then you can enter the expressions for the roots in terms of **A**, **B**, and **C**. So you punch in

2.345 SHIFT RCL A , 6.543 SHIFT RCL B , (-) 9.876 SHIFT RCL C

(From now on we will simply write **STO** instead of **SHIFT RCL**.) If you want to check that your values have been stored properly, press **RCL A** , **RCL B** , **RCL C**. To find one of the roots, punch in

((-) ALPHA B + √ (ALPHA B x^2 - 4 ALPHA A ALPHA C)) ÷ 2 ALPHA A =

to get **1.086396861** in the display. Note that you do not need to punch in the multiplication sign, nor the parentheses around the denominator, when your multiplications involve memory locations. (You still need to put parentheses around the numerator, however.) To get the other root, change the **+** to a **-** (see the section "**Editing Expressions**" above) to get **-3.876588759** in the display. Thus, the roots, to four significant figures, are 1.086 and -3.877.

More examples of the use of memory locations are given in the section "***Solving Triangles using Memory Locations***" below.

M is for Memory

You store values into memory location **M**, and you recall values from it, in just the same way as you do with the other memory locations. However, memory location **M** has some special features. When **M** contains a non-zero value, a little **m** appears near the top left corner of the display. It will disappear if you clear **M**, that is, if you press **0 STO M** . Furthermore, there is a special key affecting **M**. At the right end of the last row of black keys, you will see a key marked **M+**, and above it, in orange or red, **M-**. This key enables you to add the value in the display to the value stored in **M**, or, if you press **SHIFT M+**, to subtract the value in the display from the value stored in **M**.

These features go back to the early days of calculators. In those days, there was only one memory location, named, not surprisingly, **M**. There were no parenthesis keys, and operations were not performed using the conventional order of operations, so that if you pressed **3 + 4 × 5 =** , you got **35**. But there were keys marked **STO** and **RCL**, which stored values to, and recalled them from, the one and only memory location, namely, **M**. If you wanted to calculate, for example, $(3 \times 4) + (5 \times 6) + (7 \times 8)$, you pressed

3 × 4 = STO 5 × 6 = M+ 7 × 8 = M+ RCL .

The **STO** operation stored the result of **3 × 4** into **M**, the first **M+** operation added the result of **5 × 6** to the result of **3 × 4** in **M**, and the second **M+** operation added the result of **7 × 8** to that. You no longer have to go through these gymnastics, but the **M+** key is still convenient in certain applications.

Example: Riemann Sums and Approximate Integration

In the old days, teachers and students rushed through the sections of the textbook on Riemann sums and approximate integration as quickly as possible, because the calculations were extremely unpleasant to do by hand. Now, however, calculators greatly simplify the work, and it is worthwhile to spend some time on this topic, both to have a clearer understanding of the basic idea of the integral, and also to learn how to get good numerical approximations to the value of

an integral when the Fundamental Theorem of Calculus cannot be applied. Textbook writers have not, by and large, caught up with the advances in technology, and so we present a fairly detailed discussion of how you can use your calculator effectively for these purposes.

We take as our basic example the integral $\int_3^6 \sqrt{x-2}\, dx$. You can easily evaluate this integral using the Fundamental Theorem of Calculus, obtaining a value of 14/3 = 4.666666… . We break the interval [3,6] into $n = 6$ subdivisions, each of width $\Delta x = (6 - 3)/6 = 0.5$. The left-hand endpoint of the interval of integration is 3.0, and the right-hand endpoint is 6.0. The intermediate subdivision points are at 3.5, 4.0, 4.5, 5.0, and 5.5. It will be convenient to write the endpoints and intermediate points as 6/2, 7/2, 8/2, 9/2, 10/2, 11/2, and 12/2. We recall that a Riemann sum is a sum of the form $\sum_{i=0}^{n} f(x_i)\, \Delta x$, where x_i is an x-value in the i-th subinterval.

You start by storing Δx in memory location **D**:

0.5 STO D .

This value will stay unchanged throughout the following examples.

Example: Find the left sum of $\int_3^6 \sqrt{x-2}\, dx$ with $n = 6$.

Solution: First clear **M** (don't forget this step!):

0 STO M .

In this case, the values of x_i are the left-hand endpoints of the subintervals, namely, 6/2, 7/2, 8/2, 9/2, 10/2, and 11/2. Punch in

√ (6 ÷ 2 - 2) × ALPHA D M+

to get, not surprisingly, **0.5** in the display. Now, press the right arrow key ($\Rightarrow$) three times, so that it is over the **6**, and replace the **6** by a **7**. (See the section "**Editing Expressions**" to see how this is done on your calculator.) Then press **M+** to get **0.612372435** in the display. This is the value of $\sqrt{7/2-2}\times 0.5$, and it has been added to the value in **M**. Similarly, replace the **7** by **8** and press **M+**, the **8** by **9** and press **M+**, the **9** by **10** (careful here!) and press **M+**, and finally the **10** by **11** and press **M+**. Now recall the value in **M**:

RCL M

and if all has gone well, you should see **4.411488382** in the display. So the left sum for $n = 6$ is 4.411488382 – not very close to 4.666… . Save this value in memory location **E** for future reference by pressing **STO E** .

Example: Find the right sum of $\int_3^6 \sqrt{x-2}\, dx$ with $n = 6$.

Solution: Again, clear **M**: **0 STO M** . This time, the values of x_i are the right-hand endpoints of the subintervals, namely 7/2, 8/2, 9/2, 10/2, 11/2, and 12/2. Punch in

√ (7 ÷ 2 - 2) × ALPHA D M+

to get **0.612372435** in your display. As before, replace the **7** by **8** and press **M+**, the **8** by **9** and press **M+**, the **9** by **10** and press **M+**, the **10** by **11** and press **M+**, and the **11** by **12** and press **M+**. Now recall the value in **M**, and if all has gone well, you should have **4.91188382** in your display. This is your right sum. Save it in memory location **F** for future reference.

Example: Find the trapezoidal sum of $\int_3^6 \sqrt{x-2}\, dx$ with $n = 6$.

Solution: Since $f(x) = \sqrt{x-2}$ is increasing on [3,6], the left sum lies entirely under the graph of the function, and the right sum lies above the graph. So a reasonable thing to do is to take their average. You can do this by punching in

(ALPHA E + ALPHA F) ÷ 2 = ,

giving **4.661488382**. Geometrically, this corresponds to the sum of the areas of the trapezoids formed by connecting the points on the graph corresponding to $x = 6/2, 7/2, \ldots, 12/2$. For this reason, this method of estimating the value of the integral is called the Trapezoid Rule. Save it in memory location **A** for future reference.

You can calculate the trapezoidal sum directly, without calculating the left and right sums first. Start by calculating the average of $f(3)\Delta x$ and $f(6)\Delta x$, and store the result in M:

(√ (3 - 2) × ALPHA D + √ (6 - 2) × ALPHA D) ÷ 2 STO M

giving **0.75**. (You could have calculated this by hand, but why bother? The important thing is that it gets stored in **M** to start the process.) Now you punch in $f(x_i)\Delta x$ for the intermediate subdivision points (not including the endpoints), that is, $x_i = 7/2, 8/2, 9/2, 10/2,$ and $11/2$:

√ (7 ÷ 2 - 2) × ALPHA D M+ .

As before, replace the **7** by **8** and press **M+**, the **8** by **9** and press **M+**, the **9** by **10** and press **M+**, the **10** by **11** and press **M+**. Stop here! Now recall the value in **M**, which should give you **4.661488382**, the same as before. Save it in memory location **A** for future reference. This is a respectable approximation to 4.666… , but we can do better.

Example: Find the midpoint sum of $\int_3^6 \sqrt{x-2}\ dx$ with $n = 6$.

Solution: To do this, take the points x_i to be the midpoints of each of the subintervals, namely 13/4, 15/4, 17/4, 19/4, 21/4, and 23/4, and repeat the process described above. You start, of course, with **0 STO M** , and continue by punching in

√ (13 ÷ 4 - 2) × ALPHA D M+ .

Then replace the **13** by **15** and press **M+**, the **15** by **17** and press **M+**, the **17** by **19** and press **M+**, the **19** by **21** and press **M+**, and finally the **21** by **23** and press **M+**. Now recall the value in **M**, which should be **4.669244675**. Store this value in memory location **B** for further reference. Note that this also is a respectable approximation to 4.666… , in fact, even closer than the approximation given by the Trapezoidal Rule.

In the old days, textbooks passed over the midpoint sum, because it generally involved messier numbers (13/4, 15/4, … instead of 6/2, 7/2, …) than the Trapezoidal Rule. Nowadays, the fact that the numbers are messier is no obstacle to your calculator. Calculating the midpoint sum is no harder that applying the Trapezoidal Rule, in fact, it is actually somewhat easier. If you go into the mathematical theory, you discover that in general the difference between the value given by the midpoint sum and the true value of the integral is about half the difference between the value given by the Trapezoidal Rule and the true value of the integral. Furthermore, when the graph of the function is concave down, as it is for $f(x) = \int_3^6 \sqrt{x-2}\ dx$, the Trapezoidal Rule gives a value less than the exact value of the integral, and the midpoint sum gives a value greater than the exact value of the integral. When the graph of the function is concave up, all the inequalities are reversed. So you can get an even better approximation to the exact value of the integral by taking a weighted average of the values given by the Trapezoidal Rule and the midpoint sum, giving the midpoint sum twice as much weight as the trapezoidal sum. This method for approximating integrals is called Simpson's Rule.

Example: Approximate $\int_3^6 \sqrt{x-2}\ dx$ with $n = 6$ using Simpson's Rule.

Solution: Since you stored the results from the Trapezoidal Rule and the midpoint sum in memory locations **A** and **B** respectively, you simply punch in

<div align="center">

(ALPHA A + 2 ALPHA B) ÷ 3 =

</div>

to get **4.666659244**. This is an excellent approximation to the true value! This is typical of results obtained by Simpson's Rule. In fact, Simpson's Rule is an industrial-strength method for approximating the values of definite integrals widely used by engineers and applied mathematicians when it is difficult or impossible to find an anti-derivative. An example of this would be $\int_3^6 \sqrt{x^3 - 2}\ dx$.

Note: What we have given above as the Simpson's Rule approximation for the integral with $n = 6$ is what some books would call the approximation for $n = 12$. What they call the approximation for $n = 6$ is the Simpson's Rule approximation obtained by our method starting from the Trapezoidal Rule and midpoint sum for $n = 3$.

To make sure you understand how to use the above method, try it on $\int_0^1 \frac{4}{1 + x^2}\ dx$ with $n = 4$. The true value of this integral is $\pi = 3.141592654\ldots$. The values given by the left sum, right sum, Trapezoidal Rule, midpoint sum, and Simpson's Rule are respectively 3.381176471, 2.881176471, 3.131176471, 3.146800518, and 3.141592502.

Play it again, Sam

Your calculator has a feature called "replay" which enables you (sometimes) to see previous expressions and their results. It is accessed using the up arrow (⇑) and down arrow (⇓) keys at the top of the keyboard. To see how it works, press

<div align="center">

3 + 4 × 5 =

</div>

to get **23.** , and

<div align="center">

6 + 7 × 8 =

</div>

to get **62.** in the display. Press the ⇑ key. You will see

<div align="center">

3 + 4 × 5 =

23.

</div>

again, with a small triangle pointing downwards (   ) at the right of the top line. If you now press ⇓ , you will see

<div align="center">

6 + 7 × 8 =

62.

</div>

once again, and there will be a small triangle pointing upwards (   ). You can flip back and forth between the expressions using ⇑ and ⇓. You can enter more expressions and flip between them. When you are at the "bottom" (most recent) expression, you will see only the triangle pointing upwards; when you are at the "top" (oldest) expression, you will see only the triangle pointing downwards; and when you are between the two, you will see both triangles.

There are certain limitations on what you can do. The first is the limitation of memory. The Casio can hold a maximum of 128 characters. When this memory is full, expressions are deleted, starting with the oldest. The second is that under certain circumstances, this memory is cleared. Not surprisingly, it is cleared by the various mode and reset operations (see "**Modes and Setup**" below). Unlike the memory locations **A**, **B**, **C**, **D**, **E**, **F**, **M**, **X**, and **Y**, it is also

cleared when the calculator is turned off (including when the calculator turns itself off automatically). It is not cleared when you press **AC**.

In the example above where we found the solutions of a quadratic equation by storing the coefficients a, b, c into memory locations **A**, **B**, **C** respectively, it would be nice to be able to change the values of these coefficients and find solutions of the resulting equation without typing the quadratic formula back into the calculator. On the your calculator, this is possible.

Example: Solve the quadratic equation $ax^2 + bx + c = 0$, where $a = 3$, $b = 4$, for each of the values $c = -5$, $c = -6$, and $c = -7$.

Solution: As before, punch in

3 STO A
4 STO B
(-) 5 STO C
((-) ALPHA B + √ (ALPHA B x^2 - 4 ALPHA A ALPHA C) ÷ 2 ALPHA A =

to get **0.786299647**. Now press

(-) 6 STO C

followed by the up arrow (⇑). You will see

(-B+√(B²-4AC))÷2A

(In order to see the rightmost part of this formula, you will have to press the left arrow key.)
Press **=** . You get **0.896805253**. Now press

(-) 7 STO C

followed by the up arrow again. Press **=** . You get **1.** Sure enough, $x = 1$ is a solution of the equation $3x^2 + 4x - 7 = 0$. To get the other solutions, change the **+** to a **-** in the quadratic formula and repeat the process for $c = -5$, $c = -6$, and $c = -7$.

Playing the Angles

In this section, we will assume, unless otherwise stated, that your calculator is set for degrees, with "**D**" appearing at the top. If this is not the case, see the section "**Degrees, Radians, Grads**" for your calculator under "**Modes and Setup**" below. We will also assume that we are dealing with a triangle ABC in which a, b, c are the lengths of the sides opposite the angles A, B, and C respectively.

Right Triangles

In this section, we will assume that the triangle ABC is a right triangle, with right angle at C. Then we have the relations $c^2 = a^2 + b^2$, $\sin A = \cos B = \dfrac{a}{c}$, $\cos A = \sin A = \dfrac{b}{c}$, $\tan A = \dfrac{a}{b}$, $\tan B = \dfrac{b}{a}$. The following examples illustrate the use of the calculator in solving for various parts of the triangle.

Example: Find c, given $a = 3.45$ m and $b = 7.65$ m.

Solution: In this case, you have $c = \sqrt{a^2 + b^2}$, so you press

√ (3 . 4 5 x^2 + 7 . 6 5 x^2) =

to get **8.391960438** . (Note the parentheses!) Rounding off to three significant figures (because the data are given with three significant figures), our answer is $c = 8.39$ m.

Example: Find a, given $c = 4.567$ cm and $B = 34.56°$.

Solution: In this case, you have $a = c \cos B$, so you press

4.567 × cos 34.56 =

to get **3.761073373** . Rounding off to four significant figures, our answer is a = 3.761 cm.

Example: Find A, given a = 3.45 m and b = 7.65 m.

Solution: In this case, you have $\tan A = \dfrac{a}{b}$. In order to solve for A, you have to "undo" the tan function. This is done by the $\tan^{-1}$ function, sometimes called the arctan function. To access this function, press **SHIFT tan**. Refer back to the section "**Keys for Functions and their Inverses**" under "**Exponentials and Logarithms**" above. As indicated in that section, we will simply write **tan⁻¹** for **2nd F tan** or **SHIFT tan**; similarly for **sin⁻¹** and **cos⁻¹**. So to find A, you press

tan⁻¹ (3.45 ÷ 7.65) =

to get **24.27444113** . (Again, note the parentheses!) Rounding off to three significant figures, our answer is A = 24.3°.

These examples can also be done using the procedures described in the section "*Converting between Rectangular and Polar Coordinates*" below.

Degrees, Minutes, Seconds

Your calculator can handle calculations involving degrees, minutes, and seconds. They are handled by the key labeled **°'''** , which is the second key from the left in the third row of black keys. To enter an angle expressed in degrees, minutes, and seconds, press this key after you have entered the degrees, again after you have entered the minutes, and a third time after you have entered the seconds (if any). One peculiarity of the Casio is that if you do not press the **°'''** key after entering the minutes or the seconds, your calculator will give you the error message "**Syntax ERROR**".

Example: In a right-angled triangle ABC, with right angle at C, find a, given c = 4.567 cm and B = 12°34′56″.

Solution: As above, you have $a = c \cos B$, so you press

4.567 × cos 12 °''' 34 °''' 56 °''' =

to get **4.457320758**, which you round off to four significant figures as 4.457 cm.

To convert an angle expressed as degrees, minutes, and seconds into an angle expressed as degrees and decimals or vice versa, you use **SHIFT °'''**. Note that at the top left of the **°'''** key you find the left arrow ←, and you find the same left arrow ← above the **ENG** key immediately below it. However, the operations of the two left arrow keys are different. The operation of the ← over the **ENG** key is discussed in the section "**Display Formats**" below.

You can also use the **°'''** key to convert an answer from degrees and decimal parts to degrees, minutes, and seconds.

Example: Find A to the nearest minute, given a = 3.456 m and b = 7.654 m.

Solution: As above, you have $\tan A = \dfrac{a}{b}$. So you punch in

tan⁻¹ (3.456 ÷ 7.654) =

to get **24.30053777**. Now press either **°'''** or **SHIFT °'''**, to get **24°18°1.94**. (Note that minutes are displayed with the same symbol as degrees.) So the answer, to the nearest minute, is A = 24°18′.

Actually, your calculator does not care whether you are thinking of degrees and minutes or of hours and minutes, as the next remarkable example shows:

Example: One day you worked from 1:43 pm to 5:21 pm. How long did you work, and how much did you earn, at $12.34 per hour?

Solution: To find out how long you worked, subtract 1:43 from 5:21 by pressing

5 °'" 21 °'" - 1 °'" 43 °'" =

giving **3°38′0″**, that is 3 hours and 38 minutes. Then multiply this by 12.34 by pressing

× 12.34 =

to get **44°50′7.2″**(in hours, minutes, and seconds!). Convert this to decimals by pressing **°'"** to get **44.83533333**. Thus, you earned $44.84 (assuming your employer rounds your pay up). Notice that your calculator does arithmetic (correctly!) on numbers expressed in degrees, minutes, and seconds.

Law of Cosines

In the old days (before 1980!), the Law of Sines was taken up first and studied in detail, followed by the Law of Cosines, and there was a tacit understanding between teachers and students that examination questions on the Law of Cosines would require only simple calculations. The reason for this was that calculations involving the Law of Sines could be done purely by logarithms (see the section "***Why e? Why Natural Logarithms?***" above), while calculations involving the Law of Cosines generally required a messy combination of hand calculation and use of logarithms unless the numbers were carefully chosen. Actually, though, the Law of Cosines is more important for applications than the Law of Sines, because of its connection with the dot (scalar) product in vector geometry. Furthermore, with the calculator, it is not significantly harder to use than the Law of Sines.

You recall that for a triangle ABC, with sides a, b, c and angles A, B, C, where we no longer assume that angle C is a right angle, the Law of Cosines can be expressed as

$$c^2 = a^2 + b^2 - 2ab \cos C \qquad \text{or as} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

with corresponding expressions for the other sides and angles.

Example: Find c, given a = 3.456 m, b = 7.654 m, and C = 23°45′.

Solution: Press

√ (3.456 x^2 + 7.654 x^2 - 2 × 3.456 × cos 23 °'" 45 °'") =

to get **4.701446953**. To four significant figures, the answer is 4.701 m.

Example: Find C to the nearest minute, given a = 2.345 cm, b = 3.456 cm, and c = 5.432 cm.

Solution: Press

cos⁻¹ ((2.345 x^2 + 3.456 x^2 - 5.432 x^2) ÷ (2 × 2.345 × 3.456)) =

to get **138.0967454**. Then press **°'"** to get **138°5′48.28″**. Rounded off to the nearest minute, the answer is 138°6′. Notice that the calculator automatically gives an obtuse angle, because cos C turns out to be negative.

Law of Sines

The Law of Sines can be expressed as $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

Example: Find a if A = 34°56′, B = 54°32′, and c = 76.54 m.

Solution: Solving for a in terms of c, A, and C, you have $a = \dfrac{c \sin A}{\sin C}$. You are not given C, but you can find it, since C = 180° - (A + B). Actually, you do not need to calculate it separately. You can do it all in one calculation, without converting the degrees and minutes to decimals:

76.54 × sin 34 °'" 56 °'" ÷ sin (180 - (34 °'" 56 °'" + 54 °'" 32 °'")) =
giving ***43.83045729***. To four significant figures, the answer is 43.83 m.

Do not use the Law of Sines to solve for angles of a triangle, because the Law of Sines cannot distinguish between an acute angle and an obtuse angle. See the discussion of this matter in the section "**Solving Triangles using Memory Locations**" below.

Converting between Degrees and Radians

Sometimes you need to convert an angle given in degrees to the same angle given in radians or vice versa. Of course you can always multiply by π/180 or 180/π, if you can remember which one to use, but your calculator can make the conversion for you.

To convert from an angle in degrees to an angle in radians, start with your calculator in **R** mode. (If necessary, press **MODE** twice and then **2**.) Then press **SHIFT Ans** (that is, **DRG**). The display will read

D	R	G
1	**2**	**3**

Press **1 =** .

Example: Convert 60° to radians.
Solution: In **R** mode, press **60 SHIFT Ans 1 =**. The display will read
60°
1.047197551
Thus, 60° equals approximately 1.047 radians. Note that your calculator stays in **R** mode.

To convert from an angle in radians to an angle in degrees, start with your calculator in **D** mode. (If necessary, press **MODE** twice and then **1**.) Then press **SHIFT Ans** (that is, **DRG**). The display will be as above. Press **2 =** .

Example: Convert 2.5 radians to degrees.
Solution: In **D** mode, press **2.5 SHIFT Ans 2 =** . The display will read
2.5^r
143.2394488
Thus, 2.5 radians is approximately 143.24°. Note that your calculator stays in **D** mode.

Solving Triangles Using Memory Locations

Given a triangle ABC, we will use memory locations **A**, **B**, and **C** to store the lengths of the three sides a, b, and c, and the memory locations **D**, **E**, and **F** to store the sizes of the three angles A, B, and C respectively. Given three of the six quantities a, b, c, A, B, C, to "solve the triangle ABC" means to find (if possible) the remaining three.

Given Three Sides ("SSS Case")

Example: Solve the triangle ABC, given a = 5.678 m, b = 6.543 m, c = 7.654 m.
Solution: First store the values of a, b, and c in memory locations **A**, **B**, and **C** respectively:

5.678 STO A ,   6.543 STO B ,   7.654 STO C

Now use the Law of Cosines to solve for *A*, *B*, and *C*. Recall that

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac}, \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

So for angle *A*, you punch in

cos^{-1} ((ALPHA B x^2 + ALPHA C x^2 - ALPHA A x^2) ÷ 2 ALPHA B ALPHA C) STO D

to get **46.33485975**. Note that you can press **STO D** at the end of your expression to have the value stored in the memory location as soon as it is calculated. Similarly, you calculate angle *B* and angle *C*, and store the values in memory locations **E** and **F** as follows:

cos^{-1} ((ALPHA A x^2 + ALPHA C x^2 - ALPHA B x^2) ÷ 2 ALPHA A ALPHA C) STO E

giving **56.46929364**, and

cos^{-1} ((ALPHA A x^2 + ALPHA B x^2 - ALPHA C x^2) ÷ 2 ALPHA A ALPHA B) STO F

giving **77.19584662**. You can check this by seeing that the sum of the angles is 180°:

ALPHA D + ALPHA E + ALPHA F =

giving exactly **180**. So to the nearest minute, the angles are *A* = 46°20′, *B* = 56°28′, *C* = 77°12′.

Given Two Sides and the Included Angle ("SAS Case")

Example: Solve the triangle ABC, given *a* = 3.456 m, *b* = 7.654 m, and *C* = 23°45′.

Solution: Store the values for *a*, *b*, and *C* in memory locations **A**, **B**, and **F** respectively. As in the example in the section "***Law of Cosines***" above, you get **4.701446953**, which you save in memory location **C**. The usual advice at this point is to use the Law of Sines to find

angles *A* and *B*. Thus, $\sin A = \dfrac{a \sin C}{c}$ and $\sin B = \dfrac{b \sin C}{c}$. So, punching in

sin^{-1} (ALPHA A sin ALPHA F ÷ ALPHA F) = STO D ,

we get **17.22088307**, and punching in

sin^{-1} (ALPHA B sin ALPHA F ÷ ALPHA F) = STO E ,

we get **40.97088307**. Unfortunately, the sum of **D**, **E**, and **F** is nowhere near 180°. What has gone wrong? We try again, using the Law of Cosines. Punching in

cos^{-1} ((ALPHA B x^2 + ALPHA C x^2 - ALPHA A x^2) ÷ 2 ALPHA B ALPHA C) ,

we get **17.22088307** as before, but punching in

cos^{-1} ((ALPHA A x^2 + ALPHA C x^2 - ALPHA B x^2) ÷ 2 ALPHA A ALPHA C) ,

we get **139.0291169**, which we store in **E**. This time the sum of **D**, **E**, and **F** is 180°. The explanation is that angle *B* is an obtuse angle, whose cosine is negative. The angles 40.97…° and 139.029…° are supplementary (add up to 180°), and their sines are equal. The inverse sine function sin^{-1} gives us only the acute angle, but the inverse cosine function cos^{-1} distinguishes between the acute and the obtuse angle. The moral of the story: **DO NOT USE THE LAW OF SINES TO CALCULATE ANGLES!** The answers to the problem are *c* = 4.701 m, *A* = 17°13′, *B* = 139°2′.

Given Two Angles and the Included Side ("ASA Case")

Example: Solve the triangle ABC, given *A* = 34°45′, *B* = 76°54′, *c* = 2.345 m.

Solution: Store the values for *A*, *B*, and *c* in memory locations **D**, **E**, and **C** respectively. You can find angle *C* from the fact that the sum of the angles is 180°:

180 - (ALPHA D + ALPHA E) STO F ,

so that angle $C = 68°21'$. Now you can find (correctly!) side a and side b using the Law of Sines:

$$\text{\textcolor{red}{\textbf{ALPHA C}} sin \textcolor{red}{\textbf{ALPHA}} D ÷ sin \textcolor{red}{\textbf{ALPHA}} F \textcolor{red}{\textbf{STO A}}}$$

gives **1.438092836**, and

$$\text{\textcolor{red}{\textbf{ALPHA C}} sin \textcolor{red}{\textbf{ALPHA}} E ÷ sin \textcolor{red}{\textbf{ALPHA}} F \textcolor{red}{\textbf{STO B}}}$$

gives **2.457325994**. You can check this by calculating c, using the Law of Cosines and your calculated values for a and b:

$$\sqrt{\ } (\textcolor{red}{\textbf{ALPHA A}}\ x^2 + \textcolor{red}{\textbf{ALPHA B}}\ x^2 - 2\ \textcolor{red}{\textbf{ALPHA A ALPHA B}}\ \cos F) =$$

gives **2.345**, as it should. So the answers are $C = 68°21'$, $a = 1.438$ m, $b = 2.457$ m.

Given Two Angles and a Side Opposite ("AAS Case")

Given two angles, you can find the third, using the fact that the sum of the angles is 180°. You can then find the remaining sides, using the Law of Sines as in the ASA Case.

Given Two Sides and an Angle Opposite ("ASS Case")

Example: Solve the triangle ABC, given $a = 4.321$ m, $b = 5.432$ m, and $A = 34°56'$.

Solution: Store the values for a, b, and A in memory locations **A**, **B**, and **D** respectively. The case of the Law of Cosines involving sides a and b and angle A is

$$a^2 = b^2 + c^2 - 2bc \cos A .$$

We are given a, b, and A; we want to solve this equation for c. We can rewrite the equation as

$$0 = c^2 - 2bc \cos A + (b^2 - a^2),$$

and think of it as a quadratic equation for the unknown c, where the coefficient of c^2 is 1, the coefficient of c is $-2b \cos A$, and the constant term is $b^2 - a^2$. Using the quadratic formula, we find

$$c = b \cos A \pm \sqrt{a^2 - (b \sin A)^2} .$$

(This formula is easy to interpret geometrically.) There will be no solutions if $a < b \sin A$, there will be exactly one positive solution if $a = b \sin A$, there will be two positive solutions if $b \sin A < a < b$, and there will be only one positive solution if $a \geq b$. Since c represents a length, only positive solutions are geometrically meaningful. In this case, you can punch in

$$\text{\textcolor{red}{\textbf{ALPHA B}}\ \cos \textcolor{red}{\textbf{ALPHA}} D + \sqrt{\ } (\textcolor{red}{\textbf{ALPHA A}}\ x^2 - (\textcolor{red}{\textbf{ALPHA B}}\ sin \textcolor{red}{\textbf{ALPHA}} D)\ x^2) =}$$

to get **7.452573925**, which you can store in **C**. Changing the **+** sign to **-**, you get **1.453938345**, which you can temporarily store in, for example, **M**. Now you use the Law of Cosines, as in the SSS and SAS cases above, to get **46.04240052** for angle B, which you store in **E**, and **99.02426614** for C, which you store in **F**. Next, you fetch the second value for c, which you stored in **M**, and move it to **C**: **RCL M STO C** . You can then calculate the angles B and C as before, getting **133.9575995** and **11.10906719** respectively, or you can observe that the second value for angle B is the supplement of the first, that is, 180° - 46.042…°, and that the second value for angle C can be obtained from the fact that the sum of the angles of the triangle must be 180°. So, in either case, the answers are $c = 7.453$, $B = 46°3'$, $C = 99°1'$, and $c = 1.454$, $B = 133°57'$, $C = 11°7'$. (Notice that in both triangles there is an obtuse angle, so that the Law of Sines does not suffice for the calculation.)

Rectangular and Polar Coordinates

Your calculator can convert coordinates of points in the plane given in rectangular (*xy*) coordinates) into polar (*rθ*) coordinates and vice versa. Observe the **Pol(** key, the second from the right in the first row of black keys, with **Rec(** above and to the left of it, and the **,** (comma) key, the second from the right in the last row of black keys. These are used in converting between rectangular and polar coordinates. The results of the calculation are stored in memory locations **E** and **F**. The following examples show how this is done.

Example: Convert the rectangular coordinates (3,-4) to polar coordinates.

Solution: Press **Pol(3 , (-) 4) =** . You will see

Pol(3,-4)
                    5.

To find the θ value, press **RCL F** . You will now see

F=
-53.13010235

You can flip back to the *r*-value by pressing **RCL E**. You now see

E=
                    5.

The *r*-value and the θ-value are stored in memory locations **E** and **F**, and can be accessed by pressing **RCL E** and **RCL F** . So the polar coordinates of the point (3,-4) are (5,-53.13°).

Example: Convert the polar coordinates (8,-60°) to rectangular coordinates.

Solution: Press **2nd F Pol(8 , (-) 60) =** , that is, **Rec(8 , (-) 60) =** . You will see

Rec(8,-60)
                    4.

To find the *y* value, press **RCL F**. You will now see

F=
-6.92820323

You can flip back to the *x*-value by pressing **RCL E** again. You now see

E=
                    4.

So the rectangular coordinates of the point (8,-60°) are (4,-6.928).

Sinh, Cosh, and All the Hype

To the left of the **sin**, **cos**, and **tan** keys, you find a key labeled **hyp**. This key gives you access to the so-called hyperbolic functions sinh, cosh, and tanh (pronounced "sinch", "cosh", and "tanch"), which play an important role in some kinds of physics and engineering problems. Actually, these functions are really nothing new; they are defined, using the exponential function, as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \ \cosh x = \frac{e^x + e^{-x}}{2}, \ \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Their names derive from the fact that they have properties analogous to, but often slightly different from, the trigonometric functions. For example, it is easy to see that

$$\tanh x = \frac{\sinh x}{\cosh x}.$$

On the other hand, it is not immediately obvious, but you can easily check, that $\cosh^2 x - \sinh^2 x = 1$ (instead of $\cos^2 x + \sin^2 x = 1$). If you are one of the relatively small number of people who need to use hyperbolic functions, you probably need to use them a lot. So it is convenient to be able to calculate sinh 1.234, for example, by punching in

hyp sin 1.234 =

to get **1.571908059**, and similarly for the other hyperbolic functions. You do not need to worry about degrees or radians; the hyperbolic functions are completely independent of the degree-radian-grad setting.

The inverses of the hyperbolic functions are also important. Just as the hyperbolic functions are expressed in terms of the exponential function, their inverses can be expressed in terms of the ln function:

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}), \ \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), \ \tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x}.$$

You obtain these, not surprisingly, using the **SHIFT** key. To get cosh^{-1} 2.345, for example, you can punch in either

hyp SHIFT cos 2.345 =

or

SHIFT hyp cos 2.345 =

to get **1.496513422**. What happens if you punch in

SHIFT hyp SHIFT cos 2.345 = ?

Does this seem reasonable?

Fractions, Proper and Improper

In the old days, we used to spend a lot of time working with fractions. There were 12 inches to a foot, 3 feet in a yard, 16 ounces in a pound, … , and all sorts of conversions between them. Now we have the metric system, where almost everything can be handled by decimals, and as a result you have probably had less contact with operations such as 2/3 + 3/4. Fortunately, your calculator can take care of such things for you. Your calculator has a key, the leftmost key in the second row of black keys, marked **a b/c**, with **d/c** in orange above, and this key used for calculations with fractions.

You can enter a fraction either as a simple ratio, such as 2/3, or as an integer followed by a ratio, such as 2 3/4. We look first at the case of simple ratios. To calculate 2/3 + 4/5, you press (on either calculator)

2 a b/c 3 + 4 a b/c 5 = .

In your display, you will see

2 3+4 5=

1 7 15

The symbol ⌐ between two expressions means the fraction formed from the two expressions. Two symbols ⌐ separating three expressions means the integer (whole number) value of the first part plus the fraction formed by the second and third parts. Thus, the result in the second line of the display means 1 7/15, which is the sum of 2/3 and 4/5. You can enter values as improper fractions (where the numerator is greater than the denominator). Thus, if you press

5 a b/c 3 + 13 a b/c 4 = ,

you will see

5 3+13 4=

4 11 12

meaning that 5/3 + 13/4 = 4 11/12. If you want to enter a fraction as an integer followed by a fractional part, you press **a b/c** between the integer and the numerator, and again between the numerator and the denominator. Thus, to repeat the above calculation as 1 2/3 + 3 1/4, you press
$$\textbf{1 a b/c 2 a b/c 3 + 3 a b/c 1 a b/c 4 = ,}$$
and again you get **4 11 12** , that is, 4 11/12.

      Your fractions do not have to be entered in lowest terms. If you press
$$\textbf{6 a b/c 9 + 16 a b/c 20 = ,}$$
you will still get **1 11 15**, just as before – your calculator will reduce the results to lowest terms.

      You can add, subtract, multiply, or divide these fractions, and your result will be expressed as an integer plus a proper fraction. You can also square and cube fractions, using the x^2 and x^3 keys. You can even take square roots and cube roots, providing that the result is a rational number. If you press, for example, $\sqrt{}$ **(16   25) =** , you will get **4 5** as the result. If you try to perform an operation whose result is not a rational number, you will get the usual decimal approximation. You can also get the decimal approximation to a result by pressing the **a b/c** key. For example, if you press (as above) **2 a b/c 3 + 4 a b/c 5 =** , you get the result **1 7 15** . If you now press **a b/c**, you will get **1.466666667**, which is the decimal approximation to 1 7/15. If you press **a b/c** once more, the display will return to **1 7 15**. If you now press **SHIFT a b/c** , that is, **d/c**, you will see the result displayed as an improper fraction, in this case **22 15**. If you press **d/c** again, the display will return to the form **a b/c**, in this case **1 7 15**. In short, you can do just about anything you like with these fractions. One warning: You have to be careful with the order of operations. If, for example, you press **4 a b/c 5 x^2 =**, the result is **4 25**, meaning that the calculator has interpreted it as **4 a b/c (5 x^2) =** . The remedy, of course, is to insert parentheses to show exactly what you want.

      You might like to try the complicated fraction calculation given in the section "**Getting Started**" above. If all goes well, you should get the answer -5 575/858, or -4865/858.

      You can set your calculator so that the roles of the **a b/c** and **d/c** keys are reversed. See "**Fraction Display**" under "**Modes and Setup**" below.

Example: Row Reduction of Matrices

      When you take Elementary Linear Algebra, you learn how to find the solutions of systems of linear equations by row-reduction of the corresponding augmented matrix. The problem is that unless the coefficients are very carefully chosen, the calculations with fractions tend to be messy, and it is easy for errors to creep in. Sometimes the fractions can be avoided by clever interchanges of rows, but these tend to obscure the simple underlying idea of the Gauss-Jordan method. However, you can use the fraction-manipulating capabilities of your calculator to enable you to follow the straight Gauss-Jordan method without getting bogged down in messy fractions. We illustrate this idea with a 2×2 system.

      Example: Solve the system
$$3x + 5y = 7$$
$$4x - 7y = 2.$$

      Solution: The augmented matrix of this system is
$$\begin{bmatrix} 3 & 5 & 7 \\ 4 & -7 & 2 \end{bmatrix}.$$

We denote the rows of this matrix by $\mathbf{R}_1$ and $\mathbf{R}_2$. The first step is to get a 1 in the first column of $\mathbf{R}_1$. You can do this by dividing $\mathbf{R}_1$ by 3. It's not too much strain to do this by hand:

$$\begin{bmatrix} 1 & 5/3 & 7/3 \\ 4 & -7 & 2 \end{bmatrix}.$$

Next, you want to get a 0 in the first column of $\mathbf{R}_2$. You do this by taking $\mathbf{R}_2$ - 4 $\times$ $\mathbf{R}_1$. It is easy enough to see that this gives 0 in the first column of $\mathbf{R}_2$ as desired, but to get the remaining entries of $\mathbf{R}_2$, it is convenient to use the fraction operations of your calculator. For the second column of $\mathbf{R}_2$, press

> **(-) 7 - 4 $\times$ 5 a b/c 3 =**

giving **-13 2 3**, that is, -13 2/3. Similarly, for the third column of $\mathbf{R}_2$, press

> **2 - 4 $\times$ 7 a b/c 3 =**

giving **-7 1 3**, that is, -7 1/3. Thus, the result of the second row operation is

$$\begin{bmatrix} 1 & 5/3 & 7/3 \\ 0 & -13\,2/3 & -7\,1/3 \end{bmatrix}.$$

Now you want to get a 1 in the second column of $\mathbf{R}_2$, by dividing $\mathbf{R}_2$ by -13 2/3. You or I would have to do this by first converting -7 1/3 and -13 2/3 into the improper fractions -22/3 and -41/3, but your calculator can do the whole thing as is:

> **± 7 a b/c 1 a b/c 3 ÷ (-) 13 a b/c 2 a b/c 3 =**

giving a result of **22 41**, that is, 22/41. So this row operation gives

$$\begin{bmatrix} 1 & 5/3 & 7/3 \\ 0 & 1 & 22/41 \end{bmatrix}.$$

Finally, you take $\mathbf{R}_1$ - 5/3 $\times$ $\mathbf{R}_2$, by punching in

> **7 a b/c 3 - 5 a b/c 3 $\times$ 22 a b/c 41 =** ,

giving **1 18 41**, that is, 1 18/41 or 59/41. So the row-reduced matrix is

$$\begin{bmatrix} 1 & 0 & 1\,18/41 \\ 0 & 1 & 22/41 \end{bmatrix},$$

and the solution of the system of equations is x = 1 18/41, y = 22/41. It is easy to check that these values do in fact satisfy the system of equations.

It is generally not worth bothering with row-reduction for a 2 $\times$ 2 system. A more realistic system is the following:

Example: Solve the system

$$3x + 4y - 5z = 9$$
$$4x - 5y + 6z = -8$$
$$-5x + 6y + 7z = 7$$

Solution: We give only the augmented matrices for each step. You can use them to check your calculations.

$$\begin{bmatrix} 3 & 4 & -5 & 9 \\ 4 & -5 & 6 & -8 \\ -5 & 6 & 7 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4/3 & -5/3 & 3 \\ 0 & -10\,1/3 & 12\,2/3 & -20 \\ 0 & 12\,2/3 & -11/3 & 22 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4/3 & -5/3 & 3 \\ 0 & 1 & -1\,7/31 & 1\,29/31 \\ 0 & 0 & 14\,6/31 & -2\,16/31 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4/3 & -5/3 & 3 \\ 0 & 1 & -1\,7/31 & 1\,29/31 \\ 0 & 0 & 1 & -39/220 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4/3 & 0 & 2\,31/44 \\ 0 & 1 & 0 & 1\,79/110 \\ 0 & 0 & 1 & -39/220 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 91/220 \\ 0 & 1 & 0 & 1\,79/110 \\ 0 & 0 & 1 & -39/220 \end{bmatrix}$$

So the solution is $x = 91/220 = 0.413636\ldots$, $y = 1\,79/110 = 1.71818\ldots$, $z = -39/220 = -0.177272\ldots$.

Let Me Count the Ways …

There are many situations in which you need to know the number of ways something can be done. Here are some simple ones: How many ways are there of ranking ten contestants in a contest? How many ways are there of selecting a president, a secretary, and a treasurer from a club having ten members? How many ways are there of selecting a committee of three members from the club having ten members? You can use your calculator to find the answers to these questions, and others like them.

For the first problem, you can reason as follows: There are 10 ways of choosing the winner; after that one is chosen, there are 9 contestants left, so there are 9 ways of choosing the second place; thus, there are 10×9 = 90 ways of choosing the first and second places. Continuing in this way, there are 10×9×8 = 720 ways of choosing the first, second, and third places, 10×9×8×7 = 5040 ways of choosing the first four places, and so on until you have chosen the first nine places, in which case there is only one contestant left to bring up the rear. So there are 10×9×8×7×6×5×4×3×2×1 = 3628800 ways of ranking the ten contestants. This function, where you multiply together all the positive integers starting with some positive integer n and going down to 3×2×1, is called the *factorial function*, and is denoted by $n!$. Thus, $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$. (You can think of the exclamation point as standing for "Wow!", because $n!$ gets big very fast as n increases.) This function is available on your calculator. You get it by pressing **SHIFT** followed by the x^{-1} key. You observe that $n!$ occurs in yellow above the key in question. Thus, to find 10!, you press

10 SHIFT x^{-1} =

to get

3628800.

From now on, we will simply indicate the key combination for the factorial by **!** . If you try
1 ! = , you will get 1, which is reasonable enough. Now try the following:

> **0 ! =** ,   **50 ! =** ,   **100 ! =** ,   **3 × 4 ! =** , **(3 × 4) ! =** ,
> **(-) 1 ! =** ,   **(-) (1) ! =** ,   **((-) 1) ! =** .

Can you figure out why your calculator gives these results?

For the second problem, you can reason exactly as we did at the start of the previous
problem. There are 10 ways of choosing the president, 9 ways of choosing the secretary (once
the president is chosen), and 8 ways of choosing the treasurer (once the president and secretary
are chosen). Thus, there are 10×9×8 = 720 ways of choosing the three officers. There are
several ways of calculating this with your calculator. One way, of course, is simply to press
10 × 9 × 8 = . In this case, the calculation is not too hard, but it would be less pleasant if we
were counting the number of ways of choosing 10 officers out of a club having 50 members. A
shorter way of calculating is to observe that

$$10 * 9 * 8 = \frac{10*9*8*7*6*5*4*3*2*1}{7*6*5*4*3*2*1} = \frac{10!}{7!} .$$

Thus, you can calculate the number by pressing **10 ! ÷ 7 ! =** . Similarly, the number of ways
of choosing 10 officers out of a club with 50 members is found by pressing **50 ! ÷ 40 ! =** ,
giving **3.727604302 × 10¹⁶** . In general, the number of ways of choosing r objects out of a set
of n objects, taking order into account, is called the *number of permutations of n objects taken r
at a time*, and is denoted by several notations, one of which is $_nP_r$. (There are other notations,
such as P_r^n , $P_{n;r}$, and P(n,r).) There is a formula for $_nP_r$, namely $_nP_r = \dfrac{n!}{(n-r)!}$, but the good
news is that there is a key on your calculator which gives the result of this calculation in one
step. On your calculator, you will find $n\mathrm{P}r$ in orange above the second key in the second row,
marked *n*C*r*. (We will discuss the operation of this key below.) Thus, to find the number of
ways of choosing 3 officers in a club with 10 members, you press

> **10 <mark>SHIFT</mark> nCr 3 =** ,

giving **720** as before. From now on, we will denote the key combinations <mark>SHIFT</mark> **nCr** by $_nP_r$.
To calculate the number of permutations of 50 objects taken 10 at a time, you press
50 $_nP_r$ 10 = , giving **3.727604302 × 10¹⁶** as before.

For the third problem, we observe that the order of the committee members does not
matter. Thus, if the committee consists of A, B, and C, it does not matter whether A is chosen
first, B second, and C third, or whether C is chosen first, A second, and B third. So to get the
number of distinct committees with three members, we have to divide by the number of ways of
arranging the three members. This gives $\dfrac{10*9*8}{3*2*1} = \dfrac{10!}{7!*3!} = 120$ ways of choosing the
committee. In general, the number of ways of choosing r objects out of a set of n objects, where
order is not taken into account, is called the *number of combinations of n objects taken r at a
time*, and is denoted by several notations, one of which is $_nC_r$. (There are other notations, such as
C_r^n, $C_{n;r}$, and C(n,r), and an old-fashioned notation $\binom{n}{r}$, which fortunately is going out of style.)

There is a formula for $_nC_r$, namely $_nC_r = \dfrac{n!}{r!(n-r)!}$, but again there is a key combination on your
calculator which will do the job in one step. On your calculator, you will find *n*C*r* on the second

key in the second row. Thus, to find the number of possible 3-member committees in a club with 10 members, you press

$$10 \ _nC_r \ 3 \ = \ ,$$

giving **120** as before.

Many problems involve the functions $_nP_r$ and $_nC_r$ together with other functions. These can, and should, be solved using a single calculation with the calculator.

Example: Find the number of 5-card poker hands drawn from a standard 52-card deck that contain exactly two aces.

Solution: A five-card hand that contains exactly two out of the four aces also contains three out of the 48 non-aces. Clearly the order in which the five cards are dealt does not matter. So the total number of hands is $_4C_2 * _{48}C_3$. We can evaluate this on the calculator just as we see it: **4 $_nC_r$ 2 × 48 $_nC_r$ 3 =** , giving 103776.

We can also solve probability problems this way.

Example: Find the probability that a 13-card hand dealt from a standard 52-card deck contains exactly 5 spades.

Solution: We want to find what proportion of the 13-card hands drawn from a standard deck contain exactly 5 spades (and 8 non-spades). The number of possible 13-card hands drawn from a standard deck is $_{52}C_{13}$. The number of ways of choosing 5 spades from the 13 spades in the deck is $_{13}C_5$, and the number of ways of choosing 8 non-spades from the 39 non-spades is $_{39}C_8$. So the number of hands containing exactly 5 spades and 8 non-spades is $_{13}C_5 \times _{39}C_8$. Thus, the proportion of hands containing exactly 5 spades is

$$\frac{_{13}C_5 \times \, _{39}C_8}{_{52}C_{13}}.$$

You can punch this in just as you see it:

$$13 \ _nC_r \ 5 \ \times \ 39 \ _nC_r \ 8 \ \div \ 52 \ _nC_r \ 13 \ = \ ,$$

giving **0.124691925**. (You do not need to put parentheses into this calculation, although you may if you wish.)

Since your calculator has keys for $_nP_r$ and $_nC_r$, there is no reason for you to use the formulas involving factorials to evaluate these functions. If you ask, "But if I don't write any factorials or other intermediate calculations on my exam paper, how does the marker know how I got my answer?", my answer again is that you should simply write the key-strokes on your paper. Unless and until examiners specifically start asking you to write your answers in terms of factorials, you might as well use the full power of your calculator. Actually, the important thing examiners are testing for in these kinds of problems is whether you know when to use $_nP_r$ and when to use $_nC_r$. For this, you have to understand what the problem is asking, and your calculator will not help you with that.

(The title of this section is taken from Elizabeth Barrett Browning's Sonnet XLIII, which begins:

"How do I love thee? Let me count the ways."

Her answer does not involve factorials. You may want to read the poem to find out how many ways she finds. You can find it in just about any anthology of English poetry.)

Vital Statistics

In this section, we look at how to use your calculator for statistical calculations. If you are interested in these applications, you probably either are taking a statistics course, or you have

taken one and want to use what you have learned. So I assume that you are familiar with the basic concepts and notations of statistics.

Entering Statistics Mode

To enter Statistics Mode, press **MODE**. You will then see

COMP	SD	REG
1	**2**	**3**

The simplest choice is **2** for Single-variable Data. If you choose this, you will see a small raised ^{SD} at the left of the top line of the display. If you choose **3** for REGression, you will see

Lin	Log	Exp
1	**2**	**3**

with a small arrow pointing to the right. If you press **1**, your calculator will try to fit your data to a Linear relation $y = a + bx$. If you press **2**, it will try to fit the data to a Logarithmic relation $y = a + b \ln x$. If you press **3**, it will try to fit your data to an Exponential relation $y = a * x^b$. If you press none of these, but rather press the right-arrow key $\Rightarrow$, you will see

Pwr	Inv	Quad
1	**2**	**3**

with a small arrow pointing left. This time, choosing **1**, **2**, or **3** will give you a Power relation $y = a*x^b$, an Inverse relation $y = a + b/x$, or a Quadratic relation $y = a + bx + cx^2$ respectively. In the great majority of cases when you are doing regression, you will want a linear relation, so press the left-arrow key $\Leftarrow$ and then **1** for Linear regression. You will see a small raised ^{REG} at the left of the top line of the display. Whichever of these modes you choose, you will be able to access a number of Statistical-Sums and Statistical-Variables, using **SHIFT** **1** and **SHIFT** **2**. (We discuss these in detail below.) Certain keys, notably the **,** (comma) and **M+** keys, will have special properties in Single-variable Data and REGression modes. On the other hand, you will not be able to access the memory locations **A** - **F**, **X**, **Y**, and **M**.

Entering Data

THE FIRST THING TO DO, WHEN YOU START ENTERING DATA FOR A NEW PROBLEM, IS TO CLEAR OUT ANY DATA THAT MAY ALREADY BE STORED. You do this by pressing **SHIFT** **CLR** (**MODE**). You will then see

Scl	Mode	All
1	**2**	**3**

The Scl refers to Statistics clear, so press **1**. You will then see

> **Stat clear**
>
> **0.**

Now press AC, and the message Stat clear will disappear. You are ready to enter data.

The **M+** key plays a key role in entering statistical data. Below this key, you will see, in blue, the label **DT**. Henceforth we will refer to this key as the **DT** key. The simplest situation is when you are in the Single-variable Data sub-mode. Suppose your data values are 79, 93, 84, 86, 77, 63, and 46. You enter them by pressing

79	DT
93	DT
84	DT
86	DT
77	DT

        63   DT
        46   DT

DO NOT PRESS THE **=** KEY! After you enter each value, you see

        n=

               k.

where *k* is the number of data items you have entered. This helps you keep track of where you are in entering data.

      Suppose now you want to enter the data values 81, 81, 81, 81, where the value 81 is repeated four times. Of course, you could do this by entering these items one at a time, as you did above. However, an easier way is to press

        81   DT   DT   DT   DT

But what if a data item occurs 10 times or 100 times? You could, of course, press the **DT** key as many times as needed, but there is a still easier way. Above the **,** key, you will see a semi-colon **;** in yellow. In Single-variable Data and Regression modes, this key is used to data values from the frequencies associated with them, and from here on we will refer to it as the **;** key. Note that it is conveniently located next to the **DT** key. To enter the data value 73 with a frequency of 10, you press

        73   ;   10   DT

Notice that the number *k* in the display

        n=

               k.

goes up by 10. The **n=** display counts the total of the frequencies, which statisticians refer to as *n*. We will see the significance of this when we examine how to correct data.

      If you are only interested in Single-variable Data, you can skip to the section "**Correcting Data**" at this point. If you are in Regression mode, each data value consists of a pair of numbers (*x*,*y*). Go into the Linear regression sub-mode by pressing **MODE 3 1** . (This will clear out any existing data – you do not need to press **SHIFT** **CLR** **1** .) Suppose you want to enter the data pairs (79,63), (93,97), (84,87), (63,71). You press

        79   ,   63   DT
        93   ,   97   DT
        84   ,   87   DT
        63   ,   71   DT

You will see

        n=

               4.

in the display. If you want to repeat the data pair (63,71), you can just press **DT** as many times as you like. If you want to enter the data pair (70,80) with a frequency of 5, you press

        70   ,   80   ;   5   DT

Correcting Data

      Suppose you are in the Single-variable Data mode, and have entered the values 79, 93, 84, 86, 77, 63, and 46 as above. You may want to check these values over or make corrections. To do this, press the down arrow (⇓). You will see in the display

        *x*1**=**

               79.

Press ⇓ again, and you will see

Freq1=

1.

The 79 is, of course, your first data value, and the 1 is its frequency. If you do not indicate a frequency before pressing the **DT** key, your calculator assumes a frequency of 1. Pressing ⇓ repeatedly gives you

x2=

93.

Freq2=

1.

and so on. When you get to the end of the data values, in this case

Freq7=

1.

and press ⇓ again, you go back to the beginning with

x1=

79.

Similarly, you can move up the data values by pressing the up arrow (⇑).

Now suppose you want to change the 86 value to a 90. To do this, press ⇓ or ⇑ until you reach

x4=

86.

Now press

90   =   (NOT **DT**!)

and you change the X4 value to 90. Similarly, if you move to

Freq5=

1.

you can change the frequency of the X5 value (which is 77) to 3 by pressing

3   =

But suppose you want to eliminate a data item, say $x3$, entirely. To do this, you move to X3 by pressing the ⇓ and ⇑ keys. Then you press

<mark>**SHIFT**</mark> **DT**

(Note the **CL** in orange to the bottom right of the **DT** key. You can think of this as standing for CanceL data.) When you do this, the $x3$ value, with its frequency, is wiped out, and all the other data values, with their frequencies, move up in the list. Thus, you now have $x1 = 79$, Freq1 = 1, $x2 = 93$, Freq2 = 1, $x3 = 90$, Freq3 = 1, $x4 = 77$, Freq4 = 3, $x5 = 63$, Freq5 = 1, $x6 = 46$, Freq6 = 1.

If you are in REGression mode, practically everything is the same as what we have just described for Single-variable Data. The only difference is that when you run through the data values using the ⇓ and ⇑ keys, you find, for example,

x1=

79.

y1=

63.

Freq1=

1.

and so on. You can modify or delete any of these values in the same way as in the Single-variable Data case.

Displaying and Calculating with Statistical Variables

Of course, you go through this process of entering data in order for the calculator to give you various statistical quantities calculated from the data, such as the mean, standard deviation, regression equation, and correlation coefficient. You get these by pressing **SHIFT 1** and **SHIFT 2** . (Note that **S-SUM** and **S-VAR** respectively appear in orange over the **1** and **2** keys.)

Suppose first that you are in Single-variable Data mode. If you press **SHIFT 1**, you will see

Σx^2	Σx	n
1	**2**	**3**

If you press **3**, you will see

n

in the top line of the display. To get the value of n, that is, the total number of data items, taking frequencies into account, press **=**. Similarly, if you press **2**, you will see

Σx

standing for the sum of the x values (taking frequencies into account), that is, the sum of the products $x1*Freq1$, $x2*Freq2$, … , and if you press **1**, you will see

Σx^2

standing for the sum of the squares of the x values (taking frequencies into account). In each case, press **=** to see the value. Similarly, if you press **SHIFT 2**, you will see

$\bar{x}$	$x\sigma n$	$x\sigma n\text{-}1$
1	**2**	**3**

If you press **1**, you will get the mean $\bar{x} = \dfrac{\Sigma x}{n}$; if you press **2**, you will get the population

standard deviation $\sigma = \sqrt{\dfrac{\sum (x - \bar{x})^2}{n}}$; and if you press **3**, you will get the sample standard

deviation $s = \sqrt{\dfrac{\sum (x - \bar{x})^2}{n-1}}$. Again, press **=** to see the value.

Now suppose you are in REGression mode. When you press **SHIFT 1**, you will see, as before,

Σx^2	Σx	n
1	**2**	**3**

but this time you will see a small right arrow at the right of the display. If you press $\Rightarrow$, you will see

Σy^2	Σy	Σxy
1	**2**	**3**

with a small left arrow at the left of the display. By pressing **1**, **2**, or **3** for either of these displays, you will see the name of the appropriate quantity in the first line of the display, and by pressing **=**, you will see its value.

If you press **SHIFT 2** in REGression mode, you will see, as before,

$\bar{x}$	$x\sigma n$	$x\sigma n\text{-}1$
1	**2**	**3**

but again you will see a small right arrow at the right of the display. If you press $\Rightarrow$, you will see

$\bar{y}$	$y\sigma n$	$y\sigma n\text{-}1$

	1	**2**	**3**

with a small left arrow at the left of the display and a small right arrow at the right. If you press
$\Rightarrow$ again, you will see

	A	**B**	**C**
	1	**2**	**3**

and if you press $\Rightarrow$ yet again, you will see

	$\hat{x}$	$\hat{y}$
	1	**2**

this time with only a left arrow at the left of the display. Moving backwards and forwards using
$\Leftarrow$ and $\Rightarrow$, and pressing **1**, **2**, or **3** as appropriate, you will see the name of the quantity you want
in the first line of the display, and you can obtain its value by pressing **=**.
      (If you are doing a QUADratic regression, you will see

	A	**B**	r
	1	**2**	**3**

and

	$\hat{x}1$	$\hat{x}2$	$\hat{y}$
	1	**2**	**3**

instead of the last two displays.)
      Given a value of x, to find $\hat{y}$, the corresponding value of y predicted by the regression
equation, you enter the value of x . You then enter $\hat{y}$, using the last of the display screens
above, and press **=**. Similarly given a value of y, to find $\hat{x}$, the corresponding value of x
predicted by the regression equation, you enter $\hat{x}$ as above, and press **=**. In the case of quadratic
regression, there are in general two such values of x'. You get one of them by entering $\hat{x}1$ and
pressing **=** as above, and the other by entering $\hat{x}2$ and pressing **=**.
      You can use these statistical quantities in expressions. For example, you can get the
sample variance s^2 or the population variance σ^2 in the obvious way. To get s^2, you press

SHIFT 2 3 x^2 **=**

and to get σ^2, you press

SHIFT 2 2 x^2 **=**

A less obvious example is the following: Suppose you want to calculate the t-statistic

$$t = \frac{\bar{x} - \mu}{\dfrac{s}{\sqrt{n}}}$$

for a test of the hypothesis that the population mean μ equals 0.5. You can calculate this by
pressing

(SHIFT 2 1 - 0.5) ÷ (SHIFT 2 3 ÷ √ SHIFT 1 3) =

Examples

      In this section, we give examples, taken from textbooks, which will enable you to make
sure you have mastered the use of your calculator for statistical calculations. We give only the
answers.
      Problem: Given the following scores on an examination, find n, Σx, Σx^2, $\bar{x}$, s, and σ^2.

79	97	86	76	93	87	98	68	84	88	81	91

Answers: $n = 12$, $\Sigma x = 1028$, $\Sigma x^2 = 88910$, $\bar{x} = 85.7$, $s = 8.76$, $\sigma = 70.4$ (the last three rounded off to three significant figures)

Problem: A pair of dice were rolled 100 times, giving the following values:

Value	2	3	4	5	6	7	8	9	10	11	12
Freq	3	6	9	12	16	18	11	10	7	6	2

Find n, Σx, Σx^2, $\bar{x}$, s, and σ^2.

Answers: $n = 100$, $\Sigma x = 680$, $\Sigma x^2 = 5196$, $\bar{x} = 6.80$, $s = 2.40$, $\sigma^2 = 5.72$

Problem: The following data show scores on a placement test and on an examination for 10 students:

Placement	39	43	21	64	57	47	28	75	34	52
Exam	65	78	52	82	92	89	73	98	56	75

Using linear regression, find n, Σx, Σy, Σx^2, Σxy, Σy^2, $\bar{x}$, $\bar{y}$, s_x, s_y, a, b, r, and the predicted exam score for a student who got 50 on the placement test.

Answers: $n = 10$, $\Sigma x = 460$, $\Sigma y = 760$, $\Sigma x^2 = 23634$, $\Sigma xy = 36854$, $\Sigma y^2 = 59816$, $\bar{x} = 46.0$, $\bar{y} = 76.0$, $s_x = 16.6$, $s_y = 15.1$, $a = 40.8$, $b = 0.766$, $r = 0.840$, $y' = 79.1$

Modes and Setup

In this section, we look at various settings which govern how your calculator works.

Resetting

If your calculator is behaving strangely, and you can't figure out how to get it back on track, the best thing to do is to reset it to the condition it was in when you bought it. To do this, press **SHIFT CLR** . When you do this, you see in the display

Mcl	Mode	All
1	2	3

Press **3** to reset the calculator to its original state. You will see

Reset All

0.

Press **=** to clear all memories and playback (more about these in the sections "**Thanks for the Memories**" and "**Play it again, Sam**"), and you will also reset the calculator to **COMP** mode (as opposed to **SD** and **REG** modes), **Norm1** display format, **Deg** measure for angles, *a b/c* format for fractions, and **Dot** for the decimal point (as opposed to **Comma**). (These settings are all discussed in detail below.) If you change your mind and decide you really don't want to reset the calculator, press anything else (such as **AC**).

Similarly, if you do not want to reset the calculator completely, but merely erase all memories and playback, press **1** . You will see in the display

Mem clear

If you press **=**, you will clear all memories and playback. Again, if you change your mind, press **AC**.

Computational, Statistics, and Regression Modes

Your calculator can operate in three modes, **COMP** (for COMPutational), **SD** (for Statistical Data), and **REG** (for REGression). The **COMP** mode is used for everyday calculations of the kinds you have been doing above. The **SD** mode is used for statistical analysis of one variable. The **REG** mode is used for statistical analysis of the correlation

between two variables. To change modes, press the **MODE** key (second key from the right in the top row). You will see

COMP	SD	REG
1	*2*	*3*

Press *1* to go into (or stay in) **COMP** mode. If you press **2**, you will go into **SD** mode; if you press *3*, you will see a row of further choices:

Lin	Log	Exp
1	*2*	*3*

(Other choices can be reached by using the right arrow key.) If you really want to do statistics, you can refer to the section "**Vital Statistics**". If that is not what you want, press **AC** , and then press **MODE** again to get back into **COMP** mode.

Degrees, Radians, Grads

Next, you have the selection of degrees, radians, or grads. This setting only concerns the trigonometric functions sin, cos, tan, and their inverses. To change from degrees to radians, press the **MODE** key twice. You will see

Deg	Rad	Gra
1	*2*	*3*

If you press *2*, the small **D** at the top of the display changes to **R**. Similarly, if you press **MODE** twice and then *3*, the **R** changes to **G**, and if you press **MODE** twice and then *1*, the **G** changes back to **D**.

Display Formats

Your calculator can handle numbers ranging in absolute value from 1.0×10^{-99} to $9.999999999 \times 10^{99}$ (besides 0, of course). To represent these numbers on the display, it can use a variety of formats. As it comes from the store, or after you have reset it, it uses a format called **NORM1**. In this format, numbers between 0.01 (that is, 10^{-2}) and 9999999999 (that is, 10^{10} - 1) in absolute value are represented in decimal form, with 10 significant figures for numbers ≥ 1, and 9 significant figures for numbers < 1, with trailing zeroes to the right of the decimal point omitted. Thus, for example, 10/3 is displayed as *3.333333333*, 1/3 as *0.333333333*, and 1/8 as *0.125*. Numbers $\geq 10^{10}$ or < 10^{-2} are displayed in scientific notation. For example, 6.02×10^{23} is displayed as *6.02 x10²³*, with small characters **x10** to the right of the digits in the display, and the power of 10 in raised characters to the right of that. This format is probably the most useful for most purposes.

However, you may want to have your calculator display numbers in a different format. You can select other formats by pressing the **MODE** key three times, until you see

Fix	Sci	Norm
1	*2*	*3*

As we have mentioned, when the calculator is reset to its initial state, numbers are displayed in **NORM1** format. If you press *3* for **Norm**, you will see

> **Norm 1~2?**

If you press *1*, numbers will be displayed in **Norm1** format. If you press **2**, they will be displayed in **Norm2** format. The **Norm2** format is a variation of **Norm1** format in which numbers between 0.0000000001 (that is, 10^{-9}) and 9999999999 (that is, 10^{10} - 1) in absolute value are displayed in decimal. The advantage of **Norm2** format is that it may be easier to grasp the size of a number than it is in **Norm1** format. The disadvantage is that you can lose

significant digits in the display. For example, if you enter 1.2345×10^{-8} into your calculator, in **Norm1** format you will see **1.2345** $_{x10}$**$^{-08}$**, whereas in **Norm2** format you will see **0.000000012**. (The remaining significant digits are still in your calculator. If you multiply this number by 10^3, you will see 0.000012345.) You may prefer this display format. It is the default format on the Sharp.

In **Fix** format, numbers are displayed (if possible) with a specified number of decimal places. If you press the **MODE** key three times, and then select **1** for **Fix**, you will see

Fix 0~9?

You then select the number of decimal places after the decimal point which you wish to have displayed. For example, to display values in dollars and cents, you would press **2**. Then, if you enter 123.456, you will see **123.46**, with the last digit rounded up. Again, the remaining digits are still in the calculator; if you multiply this value by 10, you will see **1234.56**.

In **Sci** format, numbers are displayed in scientific notation with a specified number of significant figures. If you press the **MODE** key three times, and then select **2** for **Sci**, you will see

Sci 0~9?

If you select, for example, **4**, then the number 1234.56 will be displayed as **1.235**$_{x10}$**^{03}**. If you select **9**, it will be displayed as **1.23456000**$_{x10}$**^{03}**. As an exception to the above rule, if you select **0**, it will be displayed with 10 significant figures, thus: **1.234560000**$_{x10}$**^{03}**.

The **Eng** format is a variation of the **Sci** format in which the power of 10 is always a multiple of 3. To get **Eng** format, press the key **ENG** (the second key from the left in the last row of black keys). If you enter, for example, 12345.67, it will be displayed in scientific format as **1.234567**$_{x10}$**^{04}**. Now press **ENG**. The display changes to **12.34567**$_{x10}$**^{03}**. Engineers like this notation because it simplifies comparing numbers "in the same ball park". For example, it is not easy to see at a glance which of the numbers 2.345×10^5 and 5.432×10^4 is greater. In engineering notation, they become 234.5×10^3 and 54.32×10^3, with the same power of 10, and it is clear that the first is greater than the second.

The Casio has an interesting further twist on **Eng** format. As you observed, if you enter 12345.67 and display it in **Eng** format, you have **12.34567**$_{x10}$**^{03}**. Now, if you press **ENG** again, you see **12,345.67**$_{x10}$**^{00}**. If you press **ENG** yet again, you see **12,345,670**$_{x10}$**$^{-03}$**. Each time you press **ENG**, the decimal point moves three places to the right, and, to compensate, the power of 10 decreases by 3. Thus, the value of the number in the display is unchanged. To move the decimal point back to the left, press <mark>SHIFT</mark> **ENG**. (Note the $\leftarrow$ above the ENG key.) The display becomes **12,345.67**$_{x10}$**^{00}**. If you press <mark>SHIFT</mark> **ENG** repeatedly, the display changes to **12.34567**$_{x10}$**^{03}**, **0.01234567**$_{x10}$**^{06}**, and so on. This might be useful in a situation where you have a batch of numbers ranging between, say, 500,000 and 2,000,000. Some of them would have $\times 10^{03}$ in their expression in engineering format, and some would have $\times 10^{06}$. Using **ENG** and <mark>SHIFT</mark> **ENG**, you can see all of them with the same power of 10, and thus more easily compare them.

Fraction Display

The Casio has two more settings. To access them, press the **MODE** key four times. You will see

Disp
1

Press 1, and you will see

ab/c   **d/c**
1   *2*

These relate to the way in which the Casio handles fractions. In the default mode, which Casio labels **ab/c**, you can enter and display fractions in the form of an integer followed by a proper fraction, for example 2 3/7. You do this using the **a b/c** key, at the left of the second row of black keys. If you punch in

2 a b/c 3 a b/c 7 =,

you will see

2   3   7.

in the display. Your calculator will manipulate this as an exact fraction. For example, if you press

+ 9 a b/c 13 =,

you will see the display ***3   11   91***, meaning that 2 3/7 + 9/13 = 3 11/91. If instead you punched in

2 + 3 ÷ 7 + 9 ÷ 13 =,

you will see ***3.120879121***, and it is far from obvious that this is the decimal approximation to 3 11/91. In some cases – for example, finding solutions of linear equations with integer coefficients – it is useful to have the answers in exact fraction form. If you choose the option **d/c** by pressing *2*, you will only be able to punch in fractions in the form d/c, that is, without an integer part, and your answers will come out in the same form. Thus, instead of punching in 2 3/7 as **2 a b/c 3 a b/c 7 =**, you would have to punch it in the equivalent form 17/7, that is, as **17 a b/c 7 =**. You can find out more about using your calculator to operate with fractions in the section "**Fractions, Proper and Improper**" above.

Dot versus Comma

Finally, you can change the symbol used for the decimal point. In the English-speaking world, the decimal point is represented by the dot ("."), and the comma is used to separate thousands, millions, etc. In much of the rest of the world, just the opposite convention is used. If you want to have your calculator use the comma for the decimal point, press **MODE** four times, to get the display

Disp
1

press *1* to get the display

ab/c   **d/c**
1   *2*

as above, and then press either the right-arrow key ⇒ or the **MODE** key once more to get

Dot   **Comma**
1   *2*

and press *2* to get the comma to represent the decimal point. If now you enter the number 123456.789, you will see ***123.456,789*** in the display. To get back to the dot notation for the decimal point, repeat the above process, but press *1* at the last step.