Concordia University

Department of Mathematics and Statistics

Ph. D. Comprehensive Examination - Part A1 Stat

Date: October 2022 **Maximum Marks :** 100
Time Allowed: 3 hours **Number of Pages:** 3
Special Instructions:

- Do all questions. The marks are indicated in square brackets.
- Read the problems carefully. Show all your work and justify your answers rigourously.
- Solutions must be clearly written in the examination booklet.
- **PERMITTED: approved calculators**
- **NOT PERMITTED: books, notes, electronic devices or any material other than calculators.**

Q. 1.(**20 marks**)
(a) (**10 marks**) Let A be the 4×4 matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find the Jordan normal form of the matrix A^2.

(b)(**10 marks**) Consider the unit cube in the space R^n, *rotated* so that the vertex $(1, 1, \ldots, 1)$ is mapped to the point $A = (\sqrt{n}, 0, 0, \ldots, 0)$. Denote the origin by $O = (0, 0, \ldots, 0)$. Find the coordinates of the orthogonal projections of all the 2^n vertices of the rotated cube on the segment OA (the "space" diagonal of the cube).

Q. 2.(**20 marks**)

(a)(**10 marks**) Let f be a function in $C^1([0,\pi])$ (i.e., f and f' are both continuous) such that $f(0) = f(\pi) = 0$.

i) Prove that the integrals $I_1 = \int_0^\pi f^2(x) \frac{1}{\sin^2 x}\, dx$ and $I_2 = \int_0^\pi f'(x) f(x) \cot x \, dx$ converge.

ii) Show that $I_1 = 2I_2$.

(b)(**10 marks**) Let f be a differentiable function on an interval $[a, b]$ with $0 < a < b$. Prove that there exists a point $\theta \in (a, b)$ such that

$$\frac{\begin{vmatrix} a & b \\ f(a) & f(b) \end{vmatrix}}{a - b} = f(\theta) - \theta f'(\theta)$$

Q. 3.(**15 marks**) m red balls and n black balls are placed in a box and they are taken out one by one randomly without replacement.

(a) Let X be the total number of red balls taken out before the first black ball is removed. Find $E[X]$. Show your work.

(b) Let Y be the total number of red balls taken out before the second black ball is removed. Find $E[Y]$. Show your work.

Q. 4.(**15 marks**) Suppose that Y_1 and Y_2 are independent random variables, both having exponential distribution with mean 3. Let $X = Y_1 + Y_2$ and $Y = Y_1/Y_2$.

(a) Find the joint density function of (X, Y) and the respective marginal densities.

(b) Are X and Y independent? Why?

Q. 5. (**15 marks**) Let X_1, X_2 be iid with Exponential (θ) distribution, i.e., with common pdf

$$f(x|\theta) = \frac{1}{\theta} \exp\left(-x/\theta\right), \ x \geq 0,$$

where $\theta > 0$ is an unknown parameter. Consider the hypotheses $H_0 : \theta = 2$ vs. $H_1 : \theta = 1$, and the following test:

$$\text{reject } H_0 \text{ if } \frac{\prod_{i=1}^{2} f(X_i|2)}{\prod_{i=1}^{2} f(X_i|1)} \leq \frac{1}{2}.$$

Calculate the *size* and the *power* of the test, using the fact that $Y := X_1 + X_2$ has Gamma $(2, \theta)$ distribution, i.e., has pdf

$$g(y|\theta) = \frac{1}{\theta^2} y \exp\left(-y/\theta\right), \ y \geq 0.$$

Evaluate the integrals involved completely.

Q. 6. (**15 marks**) Consider the linear regression model:

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i, \ 1 \leq i \leq n, \ n \geq 3,$$

where $\epsilon_1, \ldots, \epsilon_n$ are iid Normal $(0, \sigma^2)$, $\beta_0, \beta_1, \sigma^2 > 0$ are unknown model parameters, but x_i, $1 \leq i \leq n$, are non-random co-variates such that $\sum_{i=1}^{n}(x_i - \bar{x})^2 > 0$, where $\bar{x} = \sum_{i=1}^{n} x_i/n$. Let $\hat{\beta}_0, \ \hat{\beta}_1$ be the *least-squares estimators* of β_0, β_1. Write down the expressions for $\hat{\beta}_0, \ \hat{\beta}_1$, and find their expectations, variances and covariance.

Good Luck!!