Department of Mathematics & Statistics
Concordia University

MAST 699 (MAST 833), Sec H
An introduction to Shimura varieties

Topics in Number Theory and Geometry
Winter 2023

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Office Hours: By appointment.

Class Schedule: Tuesdays 8:45-11:30.

Prerequisites: A good knowledge in Riemann Surfaces, Algebraic Number Theory and Algebraic Geometry is necessary.

Textbook: No required textbook. Milne's *Introduction to Shimura varieties* and Kai-Wen Lan's *An example-based introduction to Shimura varieties* will be the main references.

Evaluation: Assignments 75%, Oral Presentation 25%.

Topics: The course will be an introduction to Shimura varieties. It will cover foundational topics such as the notions of Hermitian symmetric domains, variations of Hodge structures, Shimura data, canonical models of Shimura varieties, the Eichler-Shimura isomorphism, Matsushima's formula, the L^2-cohomology of Siegel Shimura varieties.

Academic Integrity and the Academic Code of Conduct

This course is governed by Concordia University's policies on Academic Integrity and the Academic Code of Conduct as set forth in the Undergraduate Calendar and the Graduate Calendar. Students are expected to familiarize themselves with these policies and conduct themselves accordingly. "Concordia University has several resources available to students to better understand and uphold academic integrity. Concordia's website on academic integrity can be found at the following address, which also includes links to each Faculty and the School of Graduate Studies: concordia.ca/students/academic-integrity." *[Undergraduate Calendar, Sec 17.10.2]*

AN INTRODUCTION TO SHIMURA VARIETIES

Shimura varieties emerged as a fertile ground for constructing class fields and now play a crucial role in Number Theory, providing a bridge between automorphic forms and Galois representations. This course will be an introduction to Shimura varieties. We will cover foundational topics such as the notions of Hermitian symmetric domains, variations of Hodge structures, Shimura data, and canonical models of Shimura varieties. Special emphasis will be put on discussing various examples. If time permits, we will discuss (some of) the following: the Eichler-Shimura isomorphism, Matsushima's formula, and the L^2-cohomology of Siegel Shimura varieties.

The students will be evaluated by weekly graded assignments and with an oral presentation at the end of the course. Final grade: 75% assignments, 25% final oral presentation.

The main references are [4] and [3]; some helpful references for some of the background material are [2], [1].

Here is an outline of the contents of the course.

Week 1.

- Introduction to the topic: what is a Shimura variety?
- Background on Lie groups and Lie algebras I.

Week 2.

- Background on Lie groups and Lie algebras II.
- Hermitian symmetric domains: definitions.

Week 3.

- Hermitian symmetric domains: Cartan involutions.
- Classification of Hermitian symmetric domains I.

Week 4.

- Classification of Hermitian symmetric domains II.
- Detailed discussion of some "classical" examples.

Week 5.

- Recollection on algebraic groups.
- Arithmetic and congruence subgroups.

Week 6.

- The theorem of Baily and Borel
- Locally symmetric varieties.

Week 7.

- Hodge structures and their variations.
- Connected Shimura data and the associated Shimura variety.

Week 8.

- Adeles and the strong approximation theorem.
- Shimura varieties and their connected components.

Week 9.

- Modular curves and their moduli interpretations.
- Siegel modular varieties and their moduli interpretations I.

Week 10.

- Siegel modular varieties and their moduli interpretations II.
- Shimura varieties of Hodge type and their moduli interpretation.

Week 11.

- Reflex fields and definition of canonical models of Shimura varieties.
- Existence of canonical models.

Week 12.

- Siegel modular varieties as fine moduli spaces.
- Compact examples: Shimura curves and quaternionic generalisations.

Week 13.

- The Eichler-Shimura isomorphism and Matsushima's formula I.
- The Eichler-Shimura isomorphism and Matsushima's formula II.

Week 14 - Exam.

- Oral presentations of students. Possible topics are:
  (1) discussion of Hilbert-Blumenthal or Picard modular surfaces and their models,
  (2) Hermitian symmetric domains of orthogonal type or exceptional type,
  (3) Modular functions and Hilbert's twelfth problem.

References

[1] Helgason, S. *Differential Geometry, Lie Groups, and Symmetric Spaces*, vol. 34. American Mathematical Soc., 2001.
[2] Knapp, A. W. *Lie groups beyond an introduction*, vol. 140. Springer, 1996.
[3] Lan, K.-W. An example-based introduction to Shimura varieties, 2017.
[4] Milne, J. S. Introduction to Shimura varieties. *Harmonic analysis, the trace formula, and Shimura varieties 4* (2005), 265–378.