**CONCORDIA UNIVERSITY** **Department of Mathematics and Statistics**

# *PH. D. COMPREHENSIVE PART A GENERAL*

| Date | Time | Pages |
|---|---|---|
| **May 2012** | **3 hours** | **3** |

**Special Instructions:** Calculators permitted. Lined paper booklets.
**Directions:** Answer all 6 questions. Each problem is worth 10 marks.

*READ THE QUESTIONS CAREFULLY !!! SHOW ALL WORK !!!*
*JUSTIFY ALL STEPS !!! GOOD LUCK !!!*

**Problem 1 :** **(a)** Prove that if $a+b=c$, then $\max\{s_1 a, s_2 b\} \geq \frac{1}{\left(\frac{1}{s_1}+\frac{1}{s_2}\right)} c$, for any $a, b \geq 0$, $s_1, s_2 > 1$.

**(b)** We define the harmonic mean $H$ and the geometric mean $G$ as follows:

$$H(a,b) = \frac{2}{\frac{1}{a}+\frac{1}{b}} \quad , \quad G(a,b) = \sqrt{a \cdot b} \quad , \ a, b > 0 \quad .$$

Prove that

$$H \leq G \ .$$

**(c)** Let $H, G$ be as above. Let $a_0 = 1$, $b_0 = 2$ and $a_{n+1} = H(a_n, b_n)$, $b_{n+1} = G(a_n, b_n)$, $n = 0, 1, 2, \ldots$. Prove that both sequences $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n \geq 0}$ are convergent and have the same limit. Can you find the limit?

**Problem 2 :** **(a)** Let $I$ denote the identity matrix. For square matrices $A, B$, prove that if the matrix $I - AB$ is invertible, then the matrix $I - BA$ is also invertible.

**(b)** Let $S$ and $T$ be linear subspaces of $\mathbb{R}^n$. Prove: If

$$\dim(\mathrm{Span}(S \cup T)) = \dim(S \cap T) + 1 \ ,$$

then one of the subspaces is a subset of the other. Span($W$) is the smallest linear subspace containing set $W$.

**(c)** Let $\mathcal{M}^n$ denote the set of all $n \times n$ real matrices. Let $A \in \mathcal{M}^n$ be of rank $0 \leq k \leq n$. Let $\mathcal{L} = \{B \in \mathcal{M}^n : BA = 0\}$ and $\mathcal{R} = \{C \in \mathcal{M}^n : AC = 0\}$. Show that $\mathcal{L}$ and $\mathcal{R}$ are linear spaces and compute their dimensions.

**(d)** Compute the value of the determinant of the $3 \times 3$ complex matrix $A$, provided that $\mathrm{tr}(A) = 1$, $\mathrm{tr}(A^2) = -3$, $\mathrm{tr}(A^3) = 4$. [Here $\mathrm{tr}(A)$ denotes the the trace, that is, the sum of the diagonal entries of the matrix $A$.]

**Problem 3 :** Let $f : X \to Y$ be a function between two metric spaces. Are the following statements true or false? Give a sketch of a proof or a counterexample.

**(a)** If $f$ is continuous and $f(K)$ is complete in $Y$, then $K$ is complete in $X$.

**(b)** If $f$ is continuous and $K$ is complete in $X$, then $f(K)$ is complete in $Y$.

**(c)** If $X$ is compact and $f$ is continuous, then for any open $U \subset X$ the image $f(U) \subset Y$ is also open.

**(d)** If $f$ is continuous and $f(G) \subset Y$ is dense in $Y$, then $(G) \subset X$ is also dense in $X$.

**Problem 4 :**

**(a)** Prove: If both $f = u + iv$ is analytic in an open neighbourhood of $z_0$ and $f'(z_0) \neq 0$, then the lines $u =$ const and $v =$ const are perpendicular at $z_0$.

**(b) (i)** For $A > 0$ show

$$\int_0^{\pi} \exp(-A \sin t)dt < \pi \frac{1}{A} .$$

Hint: Show first that $t \in (0, \pi/2)$ implies $\sin t > 2t/\pi$.

**(ii)** Prove

$$\int_0^{+\infty} \frac{\cos x}{x^2 + a^2} dx = \frac{\pi}{2a} e^{-a} \quad , \quad a > 0 \quad .$$

**Problem 5 :**

Let us define the space $X = \{z \in \mathbb{C} : |z| \geq 1\}$ and a function $d(z_1, z_2)$ as follows:
(i) if $\arg z_1 = \arg z_2$, then $d(z_1, z_2) = |z_1 - z_2|$;
(ii) if $\arg z_1 \neq \arg z_2$, then
$d(z_1, z_2) = |z_1| - 1 + \left(\text{length of the shorter arc between } \frac{z_1}{|z_1|} \text{ and } \frac{z_2}{|z_2|}\right) + |z_2| - 1.$

**(a)** Show that $d$ defines metric on $X$ (for triangle inequality you can consider just one case of possible positioning of $z_1, z_2, z_3$);

**(b)** Sketch the balls: $B(3, 1)$, $B(i, 1)$, $B(3/2, 1)$;

**(c)** Is $(X, d)$ a complete metric space?

**(d)** Is $(X, d)$ a compact metric space?

**(e)** Is $(X, d)$ a separable metric space?

**(f)** Let $\rho$ be the standard Euclidean metric on $X$. Is the identity a continuous map from $(X, \rho)$ onto $(X, d)$ ?

**Problem 6 :** Consider the measure space $\{\mathbb{R}, \mathcal{L}, m\}$, where $m$ is Lebesgue measure.

**(a)** State Fatou's lemma.

**(b)** Let $f \in L^1(\mathbb{R}, \mathcal{L}, m)$. Prove that

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{B\in\mathcal{L}} \; m(B) < \delta \Rightarrow \int_B |f| dm < \varepsilon \; ,$$

i.e., Lebesgue integral of $f$ is absolutely continuous with respect to Lebesgue measure $m$.

**(c)** Assume that $f_n, g_n \in L^1(\mathbb{R}, m)$, $f_n \to 0$ and $g_n \to 0$ almost everywhere, as $n \to +\infty$.

Prove

$$\lim_{n\to+\infty} \int_A \frac{2f_n(x)g_n(x)}{1 + f_n^2(x) + g_n^2(x)} dm(x) = 0 \; ,$$

for any set $A \subset \mathbb{R}$ of finite measure. Show by an example that this does not extend to the whole $\mathbb{R}$.