PH. D. COMPREHENSIVE PART A GENERAL

Date	Time	Pages
June 2013	**3 hours**	**3**

Special Instructions: Calculators permitted. Lined paper booklets.
Directions: Answer all 6 questions. Each problem is worth 10 marks.

READ THE QUESTIONS CAREFULLY !!! SHOW ALL WORK !!!
JUSTIFY ALL STEPS !!! GOOD LUCK !!!

Problem 1 : Consider the measure space $\{\mathbb{R}, \mathcal{L}, m\}$, where m is Lebesgue measure.

(a) State Lebesgue Monotone Convergence theorem.

(b) Let $\{f_n\}$ be a sequence of measurable functions satisfying $|f_n| \leq g$ almost everywhere for all $n \geq 1$, where g is an integrable function. Prove:

$$\int_{\mathbb{R}} \liminf_{n\to\infty} f_n dm \leq \liminf_{n\to\infty} \int_{\mathbb{R}} f_n dm \ .$$

(c) Let $f : [0, +\infty) \to \mathbb{R}$ be a Lebesgue integrable function such that $\int_0^t f dm = 0$ for all $t \geq 0$. Prove that $f = 0$ almost everywhere.

(d) Calculate

$$\lim_{n\to+\infty} \int_0^{+\infty} \left(\frac{\sin^n(x^2)}{x^2} \right) dm(x) \ ,$$

if it exists.

Problem 2 : Let (X, d) be a metric space.

(a) Prove:

$$|d(x, y) - d(z, w)| \leq d(x, z) + d(y, w) \ .$$

(b) Prove: If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X, then the sequence $\{d(x_n, y_n)\}$ converges in $\mathbb{R}$.

(c) If (X, d) is compact, then it is complete.

(d) If (X, d) is compact, then it is separable (there exists a countable dense subset).

Problem 3 : **(a)** Let the $(n+m)\times(n+m)$ matrix M satisfy condition:

$$m_{i,j}=0\ ,\ \text{for } 1\le i,j\le n \text{ and } n+1\le i,j\le n+m\ .$$

Prove that if λ is an eigenvalue of M, then $-\lambda$ also is the eigenvalue of M.

Examples:

$$M_1=\begin{bmatrix}0 & a & b\\ c & 0 & 0\\ d & 0 & 0\end{bmatrix}\ ,\qquad M_2=\begin{bmatrix}0 & 0 & a & b\\ 0 & 0 & c & d\\ e & f & 0 & 0\\ g & h & 0 & 0\end{bmatrix}\ .$$

(b)

(b1) Prove or disprove: If two 5×5 matrices have the same characteristic polynomial and the same minimal polynomial, they have to be similar.

(b2) Prove or disprove: (i) the set V of real valued differentiable functions defined on the reals form a vector space over the reals.

(ii) the derivative is a linear transformation from V to V.

(b3) True or false: if A is a 3×3 matrix that has three different eigenvalues, then A is diagonalizable.

(b4) True or false: if the set $\{u,v,w\}$ is linearly independent, then so is

$$\{u,u-v,u+v+w\}\ .$$

(b5) Let V be the set of real 2×2 matrices. Let A be an element of V. Decide whether or not the map $B\mapsto AB$ is a linear operator on V.

(b6) True of false: A linear transformation $T:V\to V$ that is onto, must be an isomorphism.

(c) An $n\times n$ matrix A has the property: each row contains only two non-zero elements, one on the diagonal which is larger than 1 and another outside the diagonal equal to 1. Can A be singular?

Problem 4 : **(a)** Let (X,d) be a metric space. For any two $A,B\subset X$ we define

$$D(A,B)=\inf_{x\in A,y\in B} d(x,y)\ .$$

Prove that X is compact if and only if $D(A,B)>0$ for any two closed disjoint subsets of X.

(b) Let us consider metric space $(\mathbb{N},d)$ with the metric (do not prove it is a metric):

$$d(n,m)=\begin{cases}1+\frac{1}{n+m} & ,\ \text{for } n\neq m\ ;\\ 0 & ,\ \text{for } n=m\ .\end{cases}$$

(b1) Prove that X is complete.

(b2) Consider closed balls $B_n=B(n,1+\frac{1}{2n})$. Show that they form a decreasing sequence of sets ($B_{n+1}\subset B_n$) with empty intersection.

Problem 5 : **(a)** We know that $x+e^x = y+e^y$. Does this imply that $\sin x = \sin y$?

(b) Let $f : \mathbb{R} \to \mathbb{R}$ satisfy: $\lim_{x\to\infty} f(x) = c$ and $\lim_{R\to\infty} \frac{1}{R}\int_0^R f(t)dt = 2013$. Prove that $c = 2013$.

If we change 2013 to 0, would this imply that $c = 0$?

(c) Function $f : [0,\pi] \to [0,1]$ is continuous. Show that there exists an $x_0 \in [0,\pi]$ such that $f(x_0) = \sin(x_0)$.

(d1) Let us assume that

$$\lim_{n\to\infty} ((a_1+1)(a_2+1)\cdots(a_n+1)) = g \ , \ 0 < g \le +\infty \ .$$

Prove that

$$\sum_{n=1}^{\infty} \frac{a_n}{(a_1+1)(a_2+1)\cdots(a_n+1)} = 1 - \frac{1}{g} \ .$$

Hint: Use the trick similar to that used in calculating the sum $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$, i.e., representing $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

(d2) Calculate

$$\sum_{n=1}^{\infty} \frac{n-1}{n!} \ .$$

Problem 6 :

(a) Calculate the integral

$$\int_{C(0,R)} \frac{f(z)}{(z-a)(z-b)} dz \ ,$$

where $C(0,R)$ is the circle of radius R centered at the origin, $|a| < R$, $|b| < R$ and f is analytic in $\mathbb{C}$. Use it to prove Liouville's theorem : A function analytic and bounded in $\mathbb{C}$ is constant.

(b) Assume that f is analytic and not constant in the disk $K(0,R)$ of radius R centered at the origin. Define the function

$$M(r) = \sup_{|z|=r} |f(z)| \ .$$

Prove that $M(r)$ is strictly increasing on $(0,R)$.

(c) Prove that $f(z) = z^8 + 3z^3 + 7z + 5$ has exactly 2 zeros in the positive quadrant ($\Re z > 0$, $\Im z > 0$).

(d) Evaluate:

$$\int_{-\infty}^{+\infty} \frac{x \sin x}{x^2+9} dx \ .$$