PH. D. COMPREHENSIVE PART A GENERAL

Date	Time	Pages
September 2012	**3 hours**	**3**

Special Instructions: Calculators permitted. Lined paper booklets.
Directions: Answer all 6 questions. Each problem is worth 10 marks.

READ THE QUESTIONS CAREFULLY !!! SHOW ALL WORK !!!
JUSTIFY ALL STEPS !!! GOOD LUCK !!!

Problem 1 : Consider the measure space $\{\mathbb{R}, \mathcal{L}, m\}$, where m is Lebesgue measure.

(a) State Fatou's lemma.

(b) Let $\{f_n\}$ be a sequence of integrable functions satisfying $0 \leq f_{n+1} \leq fn$ almost everywhere for all $n \geq 1$. Prove: If $\int_{\mathbb{R}} f_n dm \to 0$, as $n \to \infty$, then $f = \lim_{n\to\infty} f_n$ exists and is equal to 0 almost everywhere.

(c) Prove that $\int_{\mathbb{R}} |f| dm = 0$ implies $f = 0$ almost everywhere.

(d) Prove

$$\lim_{n\to+\infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dm(x) = 1 \ .$$

Problem 2 : **(a)** A matrix A is said to be skew-symmetric if $A^T = -A$ (A^T is the transposition of A). Prove: If matrix A is skew symmetric, then A^2 is symmetric and for any vector x we have $x^T A^2 x \leq 0$.

(b)

(b1) Is there a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which is represented by a diagonal matrix when written with respect to any fixed basis?

(b2) True of false: A linear transformation $T : V \to V$ that is onto, must be an isomorphism.

(b3) True or false: if A is a 3×3 matrix that has three different eigenvalues, then A is diagonalizable.

(b4) True or false: if the set $\{u, v, w\}$ is linearly independent, then so is

$$\{u, u - v, u + v + w\} \ .$$

(b5) i) True or false: Every $n \times n$ complex matrix has an eigenvalue.

(b5) ii) True or false: Every $n \times n$ real matrix has an eigenvalue.

(c) An $n \times n$ circulant matrix C takes the form

$$C = \begin{bmatrix} c_0 & c_{n-1} & \dots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & \dots & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \dots & c_1 & c_0 \end{bmatrix} .$$

Let $\omega_j = \exp(2\pi i j/n)$, $j = 0, 1, \dots, n-1$, be the n-th roots of unity. Show that the column vectors

$$v_j = \left(1, \omega_j, \omega_j^2, \dots, \omega_j^{n-1}\right) \ ,$$

$j = 0, 1, \dots, n-1$, are the eigenvectors of C and that they are linearly independent.

Problem 3 : Let $f : X \to Y$ be a function between two metric spaces.

(a) Prove that f is continuous if and only if for any $B \subset Y$

$$f^{-1}(\text{Int(B)}) \subseteq \text{Int}(f^{-1}(B)) \ ,$$

where $\text{Int}(A)$ denotes the interior of A.

(b) Is the following true: If f is continuous and K is complete in X, then $f(K)$ is complete in Y. (Justify your answer.)

(c) Is the following true: If X is compact and f is continuous, then for any closed $K \subset X$ the image $f(K) \subset Y$ is also closed. (Justify your answer.)

Problem 4 : **(a)** Let $x_0 = 2$ and $x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n}$ for $n \geq 0$. Prove that the sequence $\{x_n\}_{n \geq 0}$ is convergent and find its limit.

What happens if we change x_0 to $x_0 = 2012$?

(b) Consider the functional series

$$F(t) = \sum_{n=1}^{\infty} \frac{1}{n} \sin^n(\pi t) \quad , \quad t \in [0, 1] \ .$$

Is F well defined on $[0, 1]$?

Show that F is continuous on $[0, a]$ for any $a < 1/2$.

Is F differentiable on $[0, a]$ for any $a < 1/2$? (Justify your answer.)

(c) Function f is continuous on a circle. Show that there exists a diameter such that the values of f are equal on the diameter ends.

Problem 5 :

(a) Prove: If $f = u + iv$ is analytic in an open domain D and $|f|$ is constant in D, then f is constant.

(b) Does there exist a function f analytic in the plane satisfying $f(1/n) = 1/n^2$ for all $n \geq 1$ and $f(5) = 5$? (Justify your answer.)

(c) How many zeros has $f(z) = z^5 + 3z^3 + 7$ in the disk $D(0,2)$? How many zeros has $f(z) = z^5 + 3z^3 + 7$ in the annulus between the disks $D(0,1)$ and $D(0,2)$?

(d) Evaluate:

$$\int_{-\infty}^{+\infty} \frac{\cos x}{x^2 + 16} dx \ .$$

Problem 6 :

Let us define the space $X = \mathbb{R}^2$ and a function $d((x_1, y_1), (x_2, y_2))$ as follows:
(i) if $x_1 = x_2$, then $d((x_1, y_1), (x_2, y_2)) = |y_1 - y_2|$;
(ii) otherwise $d((x_1, y_1), (x_2, y_2)) = |x_1 - y_1| + \sqrt{2}|x_1 - x_2| + |x_2 - y_2|$.

(a) Show that d defines metric on X (for triangle inequality you can consider just one case of possible positioning of points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$);

(b) Sketch the balls: $B((3,0),1)$, $B((1,1),1)$;

(c) Is (X, d) a complete metric space? (Justify your answer.)

(d) Is (X, d) a compact metric space? (Justify your answer.)

(e) Is (X, d) a separable metric space? (Justify your answer.)

(f) Let ρ be the standard Euclidean metric on X. Is the identity a continuous map from (X, ρ) onto (X, d) ? Is the identity a continuous map from (X, d) onto (X, ρ) ? (Justify your answer.)