CONCORDIA UNIVERSITY Department of Mathematics and Statistics

PH. D. COMPREHENSIVE PART A GENERAL

Date	Time	Pages
June 2011	**3 hours**	**3**

Special Instructions: Calculators permitted. Lined paper booklets.
Directions: Answer 6 of the 8 questions at your choice. Each problem is worth 10 marks.

READ THE QUESTIONS CAREFULLY !!! SHOW ALL WORK !!! JUSTIFY ALL STEPS !!! GOOD LUCK !!!

Problem 1 : **(a)** Let

$$f(x) = a_1 x^{n_1} + a_2 x^{n_2} + a_3 x^{n_3} + a_4 x^{n_4} \ ,$$

for non-zero a_1, a_2, a_3, a_4 and pairwise different non-negative integers n_1, n_2, n_3, n_4. Show that f has at most 3 zeros in the open interval $(0, +\infty)$.

(b) Consider the series

$$S(a) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(1+a)^n} \quad , \quad a \geq 0 \ .$$

(i) Prove that $S(a)$ converges absolutely for all $a > 0$ and conditionally for $a = 0$.

(ii) Prove that $S(a)$ converges uniformly on $[0, +\infty)$.

(iii) Find explicitly $S(a)$.

Problem 2 : A matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ is called skew-symmetric if $a_{ij} = -a_{ji}$ for all $1 \leq i, j \leq n$.

(a) Show that if A is skew-symmetric and n is odd, then $\det A = 0$;

(b) Show that if A is skew-symmetric and n is even, then $\det A = \det B$, where $b_{ij} = c + a_{ij}$, for all $1 \leq i, j \leq n$ and c is a constant.

(c) Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & i \end{bmatrix} .$$

Find an unitary matrix U and a diagonal matrix D such that $U^{-1}AU = D$. (Here $i = \sqrt{-1}$.)

Problem 3 : Let $f : X \to Y$ be a function between two metric spaces. Are the following statements true or false? Give a sketch of a proof or a counterexample.

(a) If for any open set $U \subset X$ the set $f(U) \subset Y$ is also open, then f is continuous.

(b) If f is continuous and $U \subset Y$ is open, then $f^{-1}(U) \subset X$ is also open.

(c) If X is compact and f is continuous, then for any closed $F \subset X$ the image $f(F) \subset Y$ is also closed.

(d) If f is continuous and $G \subset X$ is nowhere dense, then $f(G) \subset Y$ is also nowhere dense.

Problem 4 :

(a) Prove: If both f and $\bar{f}$ are analytic in an open connected region Ω, then they are constant in Ω.

(b) Prove that equation $z^5 = \frac{1}{10}z^{10} + \frac{1}{15}z^{15}$ has exactly 5 solutions in the unit disk. Estimate the absolute value of the solution with the smallest non-zero modulus.

(c) Evaluate

$$\int_\gamma \frac{z^2+1}{z(z^2+4)}dz \quad , \quad \gamma(t) = re^{2\pi it} \quad , 0 \leq t \leq 1 \ ,$$

for **(i)** $r = 1$ and **(ii)** $r = 3$.

Problem 5 :

(a) Let $\{x_1, x_2, x_3, \ldots, x_n\}$ and $\{y_1, y_2, y_3, \ldots, y_n\}$ be two different bases of a linear space V. Show that one can find two vectors y_i and y_j of the second basis such that the collections $\{y_i, y_j, x_3, \ldots, x_n\}$ and $\{x_1, x_2, y_1, y_2, y_3, \ldots, y_n\} \setminus \{y_i, y_j\}$ are again two bases of a linear space V.

(b) Prove that if $|a_{ii}| > \sum_{k \neq i} |a_{ki}|$ for $i = 1, 2, \ldots, n$, then the matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ is invertible.

(c) Let V be an inner product vector space, and let $y, z \in V$. Define the linear operator $T : V \to V$ by $T(x) = \langle x, y \rangle z$, for all $x \in V$. Show that the adjoint operator T^* exists and find an explicit expression for it.

Problem 6 : Consider the measure space $\{\mathbb{R}, \mathcal{L}, m\}$, where m is Lebesgue measure.

(a) State Fatou's lemma.

(b) Sketch the proof of the following statement: If $f \in L^1(\mathbb{R}, m)$, then for any $\varepsilon > 0$ there exists a continuous function g vanishing outside a finite interval and such that

$$\int_{\mathbb{R}} |f - g| dm < \varepsilon \; .$$

State all the results you are using.

(c) Assume that $f_n \in L^1(\mathbb{R}, m)$ and $f_n \to 0$ almost everywhere, as $n \to +\infty$. Prove

$$\lim_{n\to+\infty} \int_{\mathbb{R}} \sin(f_n(x)) e^{-x^2} dm(x) = 0.$$

Problem 7 :

(a) Let $C^1[0,1]$ be the space of continuously differentiable functions on $[0,1]$ (at the endpoints we assume the existence of one-sided derivatives), with the norm

$$\|f\|_{C^1} = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)| \; .$$

(Do not prove that this is a norm.) Let $F : C^1[0,1] \to C^1[0,1]$ be defined by

$$F(f)(x) = \sin(x) f(x) \; .$$

Show that F is well defined and continuous.

(b) Prove that every totally bounded space is separable. Is every separable metric space also totally bounded?

Problem 8 : Let $X = \{n \in \mathbb{Z} : n \geq 1\}$ and let (n,m) denote the greatest common divisor of n and m.

(i) Prove that for any $n, m, k \in X$ we have $(n,k) \cdot (m,k) \leq k \cdot (n,m)$.

Let $d(n,m) = \log \frac{n \cdot m}{(n,m)^2}$.

(ii) Prove that (X, d) is a metric space.

(iii) Prove that (X, d) is unbounded and discrete.

(iv) Find the closed ball $\overline{B}(6, \log(4))$ with center 6 and radius $\log(4)$.

(v) Is the metric d equivalent to the standard metric $\rho(n,m) = |n - m|$?

(vi) Is the identity a continuous map between (X, d) and (X, ρ) ?