Department of Mathematics and Statistics
Concordia University
Ph. D. Comprehensive Examination – Part A General

Date: September 2010
Time Allowed: 3 hours **Number of Pages: 3**
Directions: Answer 6 of the 8 questions at your choice. Each problem is worth 10 marks.

Problem 1:

(a) Prove that if f is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b) and if $f(a) = f(b) = 0$, then for any real α there is a point $x \in (a, b)$ such that

$$\alpha f(x) + f'(x) = 0 \ .$$

(b) Show that the equation $3^x + 4^x = 5^x$ has exactly one real root.

(c) Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} e^{-nx}}{\sqrt{n + x^2}}.$$

Prove that it converges uniformly on $[0, +\infty)$. Prove that the series of moduluses converges pointwise on $(0, +\infty)$.

Problem 2:

(a) Let A be asymmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Show that

$$\lambda_k = \min_{\substack{S \\ \dim(S)=k}} \left(\max_{\substack{x \in S \\ \|x\|=1}} x^T A x \right) ,$$

where S are subspaces of $\mathbb{R}^n$.

(b) Let $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Find an orthogonal matrix U and a diagonal matrix D such that $U^{-1} A U = D$.

(c) Let V and W be finite dimensional subspaces of a vector space. Prove that

$$\dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W).$$

(Hint: start with a basis of $V \cap W$.)

Problem 3: Let $f : X \to Y$ be a function. Are the following statements true or false? Give a sketch of a proof or a counterexample.

(a) If for any close set $F \subset X$ the set $f(F) \subset Y$ is also closed, then f is continuous.

(b) If f is continuous and $F \subset X$ is closed, then $f(F) \subset Y$ is also closed.

(c) If X is compact and f is continuous, then for any closed $F \subset X$ the image $f(F) \subset Y$ is also closed.

(d) If f is continuous and $G \subset X$ is nowhere dense, then $f(G) \subset Y$ is also nowhere dense.

Problem 4:

(a) Show that the equation

$$5z^n = e^z \ , \ n \geq 1 \ ,$$

has no solutions in the annulus $1 < |z| < 2$. Show that it has at most a finite number of solutions in any horizontal strip $a < \Im z < b$ and in any vertical strip $a < \Re z < b$.

(b) Let

$$f(z) = \frac{1}{(1+z^2)^2} \ .$$

(i) What is the radius of convergence of the Taylor expansion of f centered at $z_0 = 0$. You do not have to produce the expansion.

(ii) Expand f into Laurent series centered at $z_0 = i$ (a few terms). In what domain is this expansion valid?

(iii) Use the residue theorem to evaluate

$$\int_{-\infty}^{+\infty} \frac{1}{(1+x^2)^2} dx.$$

Problem 5:

(a) Real functions f_1, f_2 are defined on interval (a, b). For any real constants c_1, c_2 the function $c_1 f_1 + c_2 f_2$ is of constant sign. Prove that f_1, f_2 are linearly dependent.

(b) How many automorphisms there are

(i) from $\mathbb{Q}$ to $\mathbb{Q}$ (field of rational numbers);

(ii) from $\mathbb{R}$ to $\mathbb{R}$ (field of real numbers);

(iii) from $\mathbb{C}$ to $\mathbb{C}$ (field of complex numbers).

(c) Are the following statements true or false? Give a sketch of a proof or a counterexample.

(i) A compact subspace of the reals is closed.

(ii) A closed subspace of the reals is compact.

(iii) A closed subspace of a compact metric space is compact.

Problem 6:

(a) Prove that the only solution to the system

$$\begin{cases} \frac{1}{2}x_1 & = a_{11}x_1 + \cdots + a_{1n}x_n \ ; \\ \frac{1}{2}x_2 & = a_{21}x_1 + \cdots + a_{2n}x_n \ ; \\ & \vdots \\ \frac{1}{2}x_n & = a_{n1}x_1 + \cdots + a_{nn}x_n \ ; \end{cases}$$

with integer coefficients $\{a_{ij}\}_{1 \le i,j \le n}$ is $x_1 = x_2 = \cdots = x_n = 0$.

(b) The elements u_1, u_2, v_1, v_2 of a group G satisfy identities

$$u_1 v_1 = v_1 u_1 = u_2 v_2 = v_2 u_2 \ ,$$

and

$$u_1^{p_1} = u_2^{p_1} = v_1^{p_2} = v_2^{p_2} = e \ ,$$

where p_1, p_2 are relatively prime positive integers. Prove that

$$u_1 = u_2 \quad \text{and} \quad v_1 = v_2 \ .$$

(d) Is the following statement true or false? Give a sketch of a proof or a counterexample.

Two matrices with the same minimal polynomial and the same characteristic polynomial are similar.

Problem 7: Consider the measure space $\{\mathbb{R}, \mathcal{L}, m\}$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of positive measurable functions.

(a) State Fatou's lemma.

(b) Prove that if $f_n \le f_{n+1}$ for all $n \ge 1$ and $f_n \to f$ a.e., then $\int f_n dm \to \int f dm$.

(c) Prove that

$$\int \sum_{n=1}^{\infty} f_n dm = \lim_{k \to \infty} \int \sum_{n=1}^{k} f_n dm.$$

(d) State Lebesgue dominated convergence theorem.

(e) Assume that $f_n \in L^1([0,1])$ and $f_n \to 0$ almost everywhere, as $n \to +\infty$. Prove

$$\lim_{n \to +\infty} \int_0^1 \sin(f_n(x)) dx = 0.$$

Is the same true if we exchange $[0,1]$ for the whole line $\mathbb{R}$?

Problem 8: Let $C[0,1]$ be the space of all continuous functions on $[0,1]$ and let $M[0,1]$ be the space of all probability measures on $[0,1]$. Let $\{f_1, f_2, f_3, \dots\}_{n=1}^{\infty}$ be a set of functions dense in the unit ball $\{f \in C[0,1] : \sup_x |f(x)| \leq 1\}$ of $C[0,1]$. (Why such a countable set exists?) For any $\mu_1, \mu_2 \in M[0,1]$ let us define

$$d(\mu_1, \mu_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} |\mu_1(f_n) - \mu_2(f_n)| \ ,$$

where $\mu_i(f) = \int_0^1 f(x) d\mu_i(x)$.

(a) Prove that $d(\cdot, \cdot)$ is a metric on $M[0,1]$. (You may need the information that if $\mu(f) = 0$ for all $f \in C[0,1]$, then $\mu = 0$. You do not have to prove this.)

(b) Prove that the convergence in metric d is equivalent to the vague (weak) convergence of measures:

$$d(\mu_n, \mu) \underset{n\to\infty}{\longrightarrow} 0 \iff \forall_{f \in C[0,1]} \ \ \mu_n(f) \underset{n\to\infty}{\longrightarrow} \mu(f) \ .$$

(d) Let us define measures μ_n by

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta(i/n) \quad , \quad n = 2, 3, 4, \dots \ ,$$

where $\delta(x)$ is Dirac's measure at x, i.e.,

$$\delta(x)(\{x\}) = 1 \ \text{ and } \ \delta(x)([0,1] \setminus \{x\}) = 0 \ .$$

Show that μ_n converge weakly to Lebesgue measure, as $n \to \infty$.

Read the problems carefully !
Show All Work !
Justify All Answers!
Good Luck !!!