Portfolio Selection with Transaction Costs and Jump-Diffusion Asset Dynamics: A Numerical Approach

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Abstract

We derive the boundaries of the region of no transaction in a two-asset portfolio selection problem of an investor with isoelastic utility and with a finite horizon when the risky asset follows a mixed jump-diffusion process in the presence of proportional transaction costs. These boundaries are shown to differ from their diffusion counterparts in relation to the jump intensity and the risk premium, as well as the investor risk aversion coefficient. We use a discretization of the continuous time distribution that converges to jump-diffusion and a general numerical approach for iid risky asset returns in discrete time. We find that our approach converges efficiently to the continuous time results in cases where these results are known. Comparative results with a recent study on the same topic are presented and it is shown that the numerical algorithm has equally attractive approximation properties to the unknown continuous time limit.

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1 Introduction

In this paper we extend the two-asset portfolio selection model under transaction costs for an investor with an isoelastic utility function and finite horizon to a jump-diffusion process for the risky asset with a jump component that may be, but is not necessarily, lognormally distributed. We introduce a discrete time approximation to the continuous time jump-diffusion process which converges weakly to it as the number of time subdivisions increases. We also introduce an efficient numerical algorithm that allows the derivation of the no transaction region of the portfolio selection model with a relatively small number of discrete time intervals. Last, we show that our results are virtually identical to those of an alternative indirect approach of the derivation of the no transaction region. In an appendix[2] we present an exact solution to a problem examined in an older well-known study of portfolio selection under simple diffusion, which had only derived an approximate solution.

We solve numerically the following problem, formulated in general terms. The investor maximizes her derived utility of consumption, be it the consumption of the entire wealth at the terminal finite date T or the consumption at all dates including the terminal date. The investor is constrained to hold two assets, a riskless bond and a risky stock, with the natural interpretation of an index. We denote the dollar holdings in the riskless bond as x and the dollar holdings in the stock as y. The investor faces proportional transaction costs at the rate k on transferring money from the stock account to the bound account and vice versa but not on liquidating her bond holdings. The choice variable of the investor at each discrete date t is the proportion of risky to riskless asset ($\lambda_t \equiv y_t / x_t$), which is a control maximizing the derived utility of consumption. When we allow for consumption at intermediate dates, the investor's problem necessitates an additional control ($\equiv c_t$), which is the optimal consumption at each discrete date t. In the discrete time setup we consider it a natural extension of our model to allow for time but not for state dependent consumption. Such an extension will not be attempted in this paper.

[2] Available from the authors on request.

In spite of its simplicity, this problem has no analytical exact or even approximate solution that we are aware of.[3] Although portfolio selection in continuous time goes back to Merton (1969, 1971), its practical implementation in the presence of transaction costs and with a finite horizon is more recent and limited almost exclusively to simple diffusion. Constantinides (1979) was the first one to prove that for an investor with an isoelastic utility and in the presence of transaction costs the optimal policy consists of a compact no transaction (NT) region in the form of a cone in which the investor does not trade, while she trades to the nearest boundary of that region if the portfolio drifts outside the region. Nonetheless, there were few attempts to implement these insights empirically, and almost all of them were for diffusion processes. Even without transaction costs the presence of jump components in the risky asset's return distribution creates technical problems that prevent exact analytical solutions to the problem of optimal portfolio choice for all but the simplest cases. The addition of transaction costs when the investment horizon is finite is also a source of difficulties even in the simple diffusion case, for which an approximate solution was not derived till 2002.

Section 2 introduces our model in both its continuous and its discrete approximation versions. Section 3 presents the numerical algorithm. Section 4 presents numerical results. Section 5 summarizes and closes. In the remainder of this section we complete a literature review of the portfolio selection rules in the presence of proportional transaction costs.

Constantinides (1979) proved an earlier conjecture in Magill and Constantinides (1976) that the NT region is a cone composed of two boundaries and the optimal investment policy is simple, i.e. it consists of trading to the closest boundary of this region if the risky to riskless asset proportion falls outside the cone formed by the two boundaries.[4] Constantinides (1986) was the first to present numerical results for the NT region. This work considered an infinite horizon problem for a diffusion process. Under a simplifying

[3] To our knowledge, the only study of portfolio selection under jump-diffusion and in the presence of transaction costs derived recently and almost concurrently with this work is Liu and Loewenstein (2008). We discuss this study further on in this section.

[4] This result was proven in fairly general settings: not necessarily Markovian risky asset returns, additively or multiplicatively separable utility, transaction costs function positively homogenous of degree one in the investment decision, possibly adapted process for the bond account, the presence of dividends, finite or infinite investment horizon. See Propositions 5 and 7 in Constantinides (1979).

assumption of state and time independent consumption, Constantinides (1986) derived the value function in a closed form; however, as the solution was composed of the value function and two first order conditions, the derivation of the NT region required numerical methods. Norman and Davies (1990) relaxed the simplifying assumption on consumption policy by considering both time and state dependent consumption and obtained a closed form solution composed of two ordinary differential equations. Their numerical results were not qualitatively different from the ones in Constantinides (1986). As opposed to Constantinides (1986) and Norman and Davies (1990), Dumas and Luciano (1991) considered a portfolio choice of an investor who maximizes the derived utility of consumption taking place upon the liquidation of the portfolio holdings at some future time *T*. Dumas and Luciano (1991) considered a limiting case as the liquidation time *T* tends to infinity. They assumed the discount factor to be endogenous to the problem, i.e. they solved for the discount factor for which the partial derivative of the value function with respect to time is zero. The results in that work differed from those of Constantinides (1986) first, in that the NT region was found to be considerably wider; second, no shift towards the riskless asset was found for increases in the transaction cost rate.

General theoretical results for portfolio selection in frictionless markets and for a wide class of jump processes were presented by Duffie *et al* (2000). Liu *at al.* (2003) extended that work and derived the Hamilton-Bellman-Jacobi (HBJ) partial differential equation[5] and the portfolio rules for such processes, but they provided numerical results only in a special (and rather unrealistic) case of jump-diffusion, for a fixed jump size. The portfolio rule was far apart from its diffusion counterpart, which is the Merton (1971) line.[6] However, this result was obtained under the condition that in the presence of a jump component the total volatility of the risky asset increases by the volatility of this component. This is a less informative comparison than when the total volatility from both the diffusion and jump components is kept constant, the case that we consider in our work.

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[5] The HBJ equation is a natural representation for the partial differential equation of the value function for the optimal investment problem.

[6] The Merton line is an optimal risky to riskless asset proportion equal to $a^*/(1-a^*)$, with $a^* = (\mu - r)/(1-\delta)\sigma^2$, the ratio of the risk premium to variance of the risky asset, for an infinite horizon and frictionless trading for an agent with power utility and relative risk aversion δ.

In the presence of transaction costs Liu and Loewenstein (2002) considered the finite horizon problem in continuous time for the two-asset simple diffusion case of Constantinides (1986). The finite horizon makes the value function time-dependent and prevents the solution of the HBJ equation. For this reason the authors replaced the fixed horizon with a random terminal date, which occurs with the n-th passage of a Poisson process.[7] Since their later work provides a solution for a jump-diffusion process, we present the Liu-Loewenstein methodology in some detail. There are two serious technical problems with solving the HBJ equation in the presence of transaction costs: First, there are two free boundaries varying through time; second, the time partial of the value function remains part of the equation for a fixed investment horizon. The Liu-Loewenstein methodology explores a randomization idea originally presented by Carr (1998) to produce an iterative sequence of ordinary differential equations whose successive solutions converge to yield the solutions for the value function and the boundaries of the no transaction region. Moreover, it was shown that the solution for a random terminal date converges to the solution for a fixed horizon equal to the expectation of this random quantity. In a later work Liu and Loewenstein (2008) used the identical formulation for the jump-diffusion case and considered the case where the jump component had a lognormal distribution, implying that the portfolio value was *not* bounded away from zero at any time.[8] They provided approximate solutions for the fixed horizon case as the convergence value of successive solutions of ordinary differential equations.[9] In a later section we demonstrate that the numerical results derived in this paper mirror the numerical results in Liu and Lowenstein (2008) for the special case that they considered. .

Our numerical methodology has certain advantages with respect to the analytical approaches. First, we are able to attack the finite-horizon problem directly, with a discrete time dynamic programming method that converges to the continuous time case as the time

[7] Under this stipulation, the horizon is Erlang distributed. Liu and Loewenstein (2002, 2008) also considered an exponentially distributed finite horizon, a case that is not relevant for the results of this paper.

[8] Framstadt *et al* (2001) also examined portfolio selection under jump-diffusion in the presence of transaction costs in the infinite horizon case but do not provide any numerical results.

[9] Liu and Loewenstein (LL, 2008) demonstrated that the solution for a random Erlang-distributed horizon converges to the solution for a fixed horizon as the number of Erlang jumps increases. On the other hand, deriving numerical results for the LL semi closed-form solution becomes difficult for a large number of Erlang jumps.

partition becomes finer. Second, since our approach admits some flexibility in modeling for the jump size distribution, we derive the solution also when the single-period jump size is bounded away from zero, which allows the investigation of cases where it is optimal to borrow in the presence of jumps. Last, the existence of an exact solution in the absence of transaction costs makes it possible to assess the accuracy of the discrete time approximation and extrapolate the results to the unknown case that we are attempting to solve.

2 Optimal Portfolio Policy under Proportional Transaction Costs

We present the dynamics in continuous time of the assets that we consider. We then consider a discrete approximation that converges to the continuous time formulation as the partition becomes finer. This allows us to formulate as a dynamic program the problem that an agent faces while undertaking investment decisions in the presence of proportional transaction costs.

2.1 Continuous time

First, we present the continuous time counterparts whose discretization we consider. The bond holdings x_t follow:

$$dx_t = rx_t dt\,, \tag{2.1}$$

where r is the continuously compounded riskless rate. Our first case is the diffusion process for the stock holdings y_t:

$$dy_t = \mu y_t dt + \sigma y_t dW_t \quad , \tag{2.2}$$

where W_t is a standard Gauss-Wiener process and μ, σ are its instantaneous mean and volatility parameters. In the second case we consider a mixed jump-diffusion process:

$$dy_t = \mu_d y_{t-} dt + \sigma_d y_{t-} dW_t + K_t y_{t-} dN_t \quad , \tag{2.3}$$

where $\mu_d = (\mu - \eta\mu_K)$ and the last term is the jump component added to the diffusion. It is assumed that the jump and diffusion components are independent. The variable K_t represents the time-t realization of the random jump size K with $K > -1$ and N is an independent Poisson counting process with intensity $\eta > 0$. The volatility of the diffusion component of the stock process σ_d is set so that the total volatility of the stock process is equal to the volatility in the pure diffusion case, which implies $\sigma_d = \sqrt{\sigma^2 - \eta\left(\sigma_K^2 + \mu_K^2\right)}$. In our numerical work we apply two assumptions for the logarithm of the jump size: first, it is distributed as $N\left(\mu_K, \sigma_K^2\right)$; second, it is binomially distributed with the same parameters.

In continuous time under the asset dynamics given by (2.1)-(2.2) the objective is the maximization of the function (2.4), where v_τ represents the optimal portfolio revisions, the additions to the stockholdings y_τ net of transaction costs

$$J(x_t, y_t, t) \equiv \max_{v_\tau \ \tau\in\{t,T\}} E_t\left[U\left(x_T + (1-k)y_T,\ T\right)\right]. \tag{2.4}$$

Stock purchases (sales) v_τ are financed by sales (purchases) of $(1+k)v_\tau$ ($(1-k)v_\tau$) of the riskless asset, where k represents the transaction cost parameter, assumed for simplicity equal for purchases and sales.[10] The asset holdings after a revision become then $y_\tau + v_\tau$ and $x_\tau - v_\tau - k\left|v_\tau\right|$.

We assume that this problem has as a solution a *simple* investment policy, characterized by a NT region defined by two investment barriers $\underline{\lambda}_\tau$ (buy boundary) and $\bar{\lambda}_\tau$ (sell boundary), $\tau \in [t,T]$ limiting the proportion $\frac{y_\tau}{x_\tau}$ of the risky to riskless asset. The investor transacts to the nearest boundary whenever $\frac{y_\tau}{x_\tau} \notin [\underline{\lambda}_\tau, \bar{\lambda}_\tau]$, while within the NT region $v_\tau = 0$ and the value function $J(x_\tau, y_\tau, \tau)$ satisfies the following HBJ equation (omitting the time subscripts)

[10] It is assumed, in line with all previous studies, that there are no transaction costs for the riskless asset.

$$J_t + rxJ_x + (\mu - \eta\mu_K)\, yJ_y + \frac{1}{2}\sigma^2 y^2 J_{yy} + \eta E[J(x, y(1+K), t^+) - J(x, y, t^-)] = 0 . \quad (2.5)$$

The boundary conditions outside the NT region are given by

$$(1+k)J_x = J_y, \ \frac{y}{x} \le \underline{\lambda}, \ (1-k)J_x = J_y, \ \frac{y}{x} \ge \overline{\lambda} \ . \quad (2.6)$$

As already noted, there are no closed form solutions for this problem, and even the fact that the optimal solution is a simple policy has not been shown rigorously to the best of our knowledge.[11] For this reason we use a "control variate" technique: we solve a discrete time problem that can be shown to converge to the continuous time solution under the asset dynamics (2.2)-(2.3) in the absence of transaction costs. For such a problem we know from Constantinides (1979) that the optimal investment policy is simple, and we derive an algorithm by which it can be found. If by setting the transaction cost parameter to zero we find that the approximation to the continuous time solution is adequate then we surmise that the same thing is true for $k > 0$.

2.2 Discretization

We consider the following discrete approximation that converges weakly to (2.3), expressed by Lemma 1 and proven in the appendix.[12]

<u>Lemma 1</u>: The following return process ($\frac{y_{t+\Delta t}}{y_t} \equiv z_{t+\Delta t}$) is a valid approximation of (2.3):[13]

$$z_{t+\Delta t} = \begin{cases} 1 + \mu_d \Delta t + \sigma_d \varepsilon \sqrt{\Delta t} & \text{with probability } 1 - \eta\Delta t \\ 1 + K & \text{with probability } \eta\Delta t \end{cases} , \quad (2.7)$$

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[11] Framstadt *et al* (2001) demonstrate it for an infinite horizon, while Liu and Lowenstein (2008) use a formulation with a random horizon.

[12] See also Oancea and Perrakis (2009).

[13] The approximation (2.4) converges weakly to (2.3), in the sense that the expectation of any continuous function of the random stock return at some future time taken with respect to the discrete process converges to the expectation of that same function taken with the continuous time limit of the process. This is the appropriate convergence criterion for our problem.

where ε is a random variable with a given distribution of mean 0 and variance 1 that can be anything. This return process is a mixture of the diffusion and jump components with corresponding probabilities $1-\eta\Delta t$ and $\eta\Delta t$.[14]

In our numerical work we use for the distribution of ε a trinomial distribution described in a later section. We lay out further details of our discretization in Section 3 and in the appendix.

Under proportional transaction cost, the bond and stock accounts dynamics are:

$$\begin{aligned} x_{t+1} &= \left(x_t - v_t - k\left|v_t\right|\right)R \\ y_{t+1} &= \left(y_t + v_t\right)z_{t+1} \end{aligned}, \tag{2.8}$$

where v_t denotes the time-t investment decision, the dollar amount net of transaction costs by which the investor changes her risky asset account. The investor solves the following problem of maximizing the expected utility of the terminal wealth net of transaction costs:

$$\max_{v_\tau \ \forall\tau\in\{t,\dots,\,T-1\}} E_t\left[U\left(x_T + (1-k)y_T,\ T\right)\right], \tag{2.9}$$

s. t. $x_t + y_t(1-k) > 0$ and $x_t + y_t(1+k) > 0$ (solvency constraints), where v_τ is the time-τ investment decision. As with the continuous time case, the solution to the investor's problem is a pair of boundaries of the NT region. We denote the buy and sell boundaries by $\underline{\lambda}_t$ and $\overline{\lambda}_t$, respectively and by λ_t the time-t risky to riskless asset proportion $\frac{y_\tau}{x_\tau}$. Note that the NT region is a convex subset of the solvency region characterized by the above two boundaries.[15] This effectively precludes borrowing for lognormally distributed jumps since the investor will face a positive likelihood of ruin.

[14] Alternatively, (2.7) can be replaced by a convolution of the diffusion and jump components. The asymptotic properties, however, remain the same. Our discrete time algorithm follows the mixture.

[15] When both x and y are positive, the solvency constraints are trivially satisfied due to limited liability. The first constraint ensures the positive net worth for borrowing, the second for selling the stock short. Under a positive risk premium and risk aversion, it is never optimal to sell short the risky asset.

The most frequently used approach to solve for (2.7) is the dynamic programming formulation:

$$V\left(x_t, y_t, t\right) = \underset{v_t}{\text{Max}}\ E_t\left[V\left(x_{t+1}, y_{t+1}, t+1\right)\right] \qquad (2.10)$$

with the boundary conditions:

$$V\left(x_T, y_T, T\right) = U\left(x_T + (1-k)\, y_T, T\right). \qquad (2.11)$$

The isoelastic utility function $\left(W_T\right)^{\alpha} / \alpha$, where $\alpha = 1-\delta$, with δ denoting the relative risk aversion (RRA) coefficient, results in a concave and homogenous of degree α in its arguments value function (2.10)-(2.11), as was shown in Constantinides (1979). This is also the approach that we use to solve the problem at hand. As we argue later, for the applied risky asset discrete-time dynamics (2.7) and the resulting state dynamics for the problem (2.10), applying (2.9) will yield an easy to apply and precise numerical solution.

A central role in our analysis will be played by two functions deriving the indirect (not necessarily maximized) utility for purchase and sale of the risky asset, respectively $\underline{J}(.)$ and $\overline{J}(.)$, which we define as:

$$\underline{J}\left(x_t, y_t, v_t, t\right) = E_t\left[V\left\{\left(x_t - (1+k)v_t\right)R, \left(y_t + v_t\right) z_{t+1}, t+1\right\}\right]$$

and . (2.12)

$$\overline{J}\left(x_t, y_t, v_t, t\right) = E_t\left[V\left\{\left(x_t - (1-k)v_t\right)R, \left(y_t + v_t\right) z_{t+1}, t+1\right\}\right]$$

To increase the proportion of the risky to riskless asset to some new proportion λ_t ($v_t > 0$), the investment decision is:

$$v_t = \frac{\lambda_t x_t - y_t}{\lambda_t (1+k) + 1}. \qquad (2.13)$$

Substituting this last quantity into the first line of (2.10) yields:

$$\underline{J}\left(x_t, y_t, \lambda_t, t\right) = \left(x_t + (1+k)y_t\right)^{\alpha} E_t\left[V\left\{\frac{R}{\lambda_t(1+k)+1}, \frac{\lambda_t z_{t+1}}{\lambda_t(1+k)+1}, t+1\right\}\right], \quad (2.14)$$

where we used the homothetic property of the value function to take the term outside the expectations operator. A similar argument for the stock sale yields:

$$\overline{J}\left(x_t, y_t, \lambda_t, t\right) = \left(x_t + (1-k)y_t\right)^{\alpha} E_t\left[V\left\{\frac{R}{\lambda_t(1-k)+1}, \frac{\lambda_t z_{t+1}}{\lambda_t(1-k)+1}, t+1\right\}\right]. \quad (2.15)$$

It can be shown that maximizing (2.14) and (2.15) with respect to λ_t yields, respectively the buy and the sell boundary of the NT region $\underline{\lambda}_t$ and $\overline{\lambda}_t$.[16] Since it is apparent that the terms in powers α are inconsequential positive quantities, we have the following program solving for $\underline{\lambda}_t$ and $\overline{\lambda}_t$:

$$\underline{\lambda}_t = \arg\max_{\lambda_t}\left[\underline{V}\left(\lambda_t, t\right)\right]$$

$$\text{and} \qquad , \qquad (2.16)$$

$$\overline{\lambda}_t = \arg\max_{\lambda_t}\left[\overline{V}\left(\lambda_t, t\right)\right]$$

where:

$$\underline{V}\left(\lambda_t, t\right) = E_t\left\{V\left(\frac{R}{\lambda_t(1+k)+1}, \frac{\lambda_t z_{t+1}}{\lambda_t(1+k)+1}, t+1\right)\right\}$$

$$\text{and} \qquad . \qquad (2.17)$$

$$\overline{V}\left(\lambda_t, t\right) = E_t\left\{V\left(\frac{R}{\lambda_t(1-k)+1}, \frac{\lambda_t z_{t+1}}{\lambda_t(1-k)+1}, t+1\right)\right\}$$

[16] An induction proof is in the Appendix to Genotte and Jung (1994), itself an application of the general result in Constantinides (1979) to the CRRA utility function.

If the NT region exists, the program (2.16)-(2.17) will always yield a solution since the value function is strictly concave.

We may now formulate compactly the investor's dynamic problem with the inclusion of the optimal investment policy:[17]

$$V\left(x_t, y_t, t\right)=\begin{cases}\left(x_t+(1+k) y_t\right)^\alpha \underline{V}\left(\underline{\lambda}_t, t\right) & \text{for } \lambda_t<\underline{\lambda}_t \\ E_t\left[V\left(x_t R, y_t z_{t+1}, t+1\right)\right] & \text{for } \underline{\lambda}_t \leq \lambda_t \leq \overline{\lambda}_t \\ \left(x_t+(1-k) y_t\right)^\alpha \overline{V}\left(\overline{\lambda}_t, t\right) & \text{for } \lambda_t>\overline{\lambda}_t\end{cases} \tag{2.18}$$

3 Numerical Analysis

We describe our numerical approach that is based on forward induction. A critical step in this approach consists of an efficient incorporation of already known solutions to the problem at all future dates. Further, we present details of the discretization of the risky asset dynamics.

3.1 Forward induction

To solve for the program (2.16)-(2.17) we apply the direct approach (2.9). This is a forward-inductive approach, that yields $\underline{V}\left(\lambda_t, t\right)$ and $\overline{V}\left(\lambda_t, t\right)$ as *continuous* functions of λ_t, resulting in flexible modeling. This admits the derivation of the NT region by reliable optimization routines resulting in highly accurate estimates. In general, for the current

[17] In an appendix (available form the authors on request) we show by using our formulation that the well-known study by Genotte and Jung (1994) for portfolio selection under diffusion contains systematic errors that may distort the results.

setting, we may write the value function as the expectation of the terminal utility of wealth with respect to the underlying probability space:[18]

$$V\left(x_t, y_t, t\right) = \max_{v_\tau \ \forall \tau\in[t,\, T-1]} \sum_j U\left(x_{T,j} + (1-k)\, y_{T,j}\right) \Pr(\omega_j)\,, \tag{3.1}$$

where ω_j represents a path j for a given discrete-time probability space, and $x_{T,j}$ and $y_{T,j}$ have been derived for the filtration of a given probability space and the optimal investment decisions at all dates $\tau = t, t+1, \ldots, T-1$. In particular, (3.1) applies to the program (2.16)-(2.17) with appropriate quantities substituted for x_t, y_t.

Since we use a (multidimensional) tree to represent the process of the risky asset, equation (3.1) appears to be difficult to solve in numerical work for more than a few time periods. The difficulty stems from the fact that the number of paths grows exponentially in time partition. To deal with this difficulty, in the following paragraphs we present a recursive model which efficiently aggregates the paths of the state variable λ_τ, $\tau = t+1, \ldots, T-1$. As we will show, exploiting the fact that the ratio $\dfrac{y_\tau}{x_\tau}$ is the sole state variable, the homothetic property of the value function and the recombination property of the assumed discretization of the risky asset dynamics will allow us to aggregate the states at each forward step.

Assume that the boundaries of the NT region $\underline{\lambda}_\tau$ and $\overline{\lambda}_\tau$ were found for all times $\tau = t+1, \ldots, T-1$. Given this information set, we define the indirect but not necessarily maximized utility for a portfolio consisting of \$1 in the riskless and \$$\lambda_t$ in the risky asset at time t for the probability space as defined in (3.1) *and* for the future optimal portfolio restructuring:

$$J\left(1, \lambda_t, t\right) \equiv \sum_j U\left(x_{T,j} + (1-k)\, y_{T,j}\right) \Pr(\omega_j) \,|\, \left\{\underline{\lambda}_\tau, \overline{\lambda}_\tau\right\}_{\tau=t+1,\ldots,T-1}. \tag{3.2}$$

[18] The idea of solving for the NT region by forward induction in numerical work was first presented in Boyle and Lin (1997). However, their numerical approach considered the first order condition on the terminal utility, which was suitable to solve the problem only for a limited number of periods.

By (2.16)-(2.17) and (3.2) it is clear that we have:

$$\underline{\lambda}_t = \arg\max_{\lambda_t} \left\{ \left[(1+k)\lambda_t + 1 \right]^{-\alpha} J(1, \lambda_t, t) \right\}$$

$$\text{and} \qquad . \tag{3.3}$$

$$\overline{\lambda}_t = \arg\max_{\lambda_t} \left\{ \left[(1-k)\lambda_t + 1 \right]^{-\alpha} J(1, \lambda_t, t) \right\}$$

We partition the time-τ paths of the state variable λ_τ into two types: the first type includes those paths which remain inside the NT region, and the second type includes the paths that fall outside this region and are traded to the nearest boundary $\underline{\lambda}_\tau$ or $\overline{\lambda}_\tau$ by virtue of the simple investment policy. The first type of paths presents no particular problem since inside the NT region the state variable will follow the recombination pattern of a given lattice. To see that, consider that in this case the time-τ portfolio holdings are $\left(R^{\tau-t}, \lambda_t \hat{Z}_\tau\right)$, with the cumulative stock return up to time τ $\hat{Z}_\tau \equiv \Pi_{i=1}^{\tau-t} z_{t+i}$ which implies $\lambda_\tau = \lambda_t \dfrac{\hat{Z}_\tau}{R^{\tau-t}}$, with the associated probabilities resulting from the ($\tau - t$)-period convolution of the one-period distribution of the risky asset with itself.

For the second type of paths, namely those for which the simple investment policy stipulates a trade to the nearest boundary of the NT region, at each time τ we derive a single number, the contribution of these paths to the terminal derived utility ($\equiv \Delta J_\tau$). As we will demonstrate, the quantity ΔJ_τ will subsume all the relevant past and future path information as of time-τ. We elaborate below on the derivation of ΔJ_τ; here we define it implicitly by

$$J(1, \lambda_t, t) = \sum_{\tau=t+1}^{T-1} \Delta J_\tau + \alpha^{-1} \sum_{l=1}^{N_T} \Pr\left(\hat{Z}_{T,l}\right) \left(R^{T-t} + (1-k)\lambda_t \hat{Z}_{T,l}\right)^\alpha, \tag{3.4}$$

where the second summation is over the N_T paths which remained inside the NT region at *each and every* time τ, $\tau = t+1,...,T-1$, with $\Pr(.)$ denoting time-t probabilities of the terminal states of the risky asset.[19]

To aggregate the time-τ paths outside the NT region to ΔJ_τ defined above, we use the homothetic property of the value function and the fact that we already solved for the value function $V\left(1,\underline{\lambda}_\tau,\tau\right)$ and $V\left(1,\overline{\lambda}_\tau,\tau\right)$, $\tau = t+1,...,T-1$ by maximizing (3.3). To demonstrate that there exists an aggregation that yields (3.2), the derived utility that subsumes all the relevant path information, we use the following result, proven in the appendix.

<u>Lemma 2</u>: The contribution ΔJ_τ of all time-τ paths outside the NT region to the time-t derived utility of terminal wealth as defined by (3.4) is the following:

$$\Delta J_\tau = \left[\underline{X}_\tau V\left(1,\underline{\lambda}_\tau,\tau\right) + \overline{X}_\tau V\left(1,\overline{\lambda}_\tau,\tau\right)\right], \tag{3.5}$$

where $\underline{X}_\tau = \sum_{i=1}^{\underline{n}_\tau} \Pr\left(\hat{Z}_{\tau,i}\right)\left(\frac{R^{\tau-t}+(1+k)\lambda_t\hat{Z}_{\tau,i}}{\underline{\lambda}_\tau(1+k)+1}\right)^\alpha$ and

$\overline{X}_\tau = \sum_{j=1}^{\overline{n}_\tau} \Pr\left(\hat{Z}_{\tau,j}\right)\left(\frac{R^{\tau-t}+(1-k)\lambda_t\hat{Z}_{\tau,j}}{\overline{\lambda}_\tau(1-k)+1}\right)^\alpha$, where $\hat{Z}_{\tau,i}$ and $\hat{Z}_{\tau,j}$ are respectively the stock returns resulting in the portfolio proportion λ_τ below or above the NT region, $\Pr(.)$ are the time-t probabilities of these returns, and we denote the time-τ number of $\hat{Z}_{\tau,i}$'s ($\hat{Z}_{\tau,j}$'s) by $\underline{n}_\tau$ ($\overline{n}_\tau$). Note that these probabilities may be derived by simply following the recombination pattern of a given lattice. The terms under the power of α are time-τ dollar values of the bond account after transacting to a given boundary of the NT region.

With the use of Lemma 2, we rewrite the maximization problem (3.3) as follows:

[19] In our numerical work, we use the fact the lattice is recombining inside the NT region, which implies that at the terminal date we have nodes rather than paths.

$$\underline{\lambda}_t = \arg\max\left\{\left[\left(1+k\right)\lambda_t+1\right]^{-\alpha}\left[\sum_{\tau=t+1}^{T-1}\Delta J_\tau+\alpha^{-1}\sum_{l=1}^{N_T}\Pr\left(\hat{Z}_{T,l}\right)\left(R^{T-t}+\left(1-k\right)\lambda_t\hat{Z}_{T,l}\right)^{\alpha}\right]\right\}$$

and , (3.6)

$$\overline{\lambda}_t = \arg\max\left\{\left[\left(1-k\right)\lambda_t+1\right]^{-\alpha}\left[\sum_{\tau=t+1}^{T-1}\Delta J_\tau+\alpha^{-1}\sum_{l=1}^{N_T}\Pr\left(\hat{Z}_{T,l}\right)\left(R^{T-t}+\left(1-k\right)\lambda_t\hat{Z}_{T,l}\right)^{\alpha}\right]\right\}$$

where the second summation is over the N_T recombined paths which remained inside the NT region at *each and every* time τ, $\tau = t+1,\ldots,T-1$.

Now we can describe the major steps of our numerical solution. Take a candidate solution λ_t for either maximization problem in (3.3) and proceed forward with the lattice. At each time τ derive the terminal contribution ΔJ_τ of all paths outside the NT region by (3.5). Delete these paths from the lattice and repeat the process till $\tau = T-1$ is reached. Since it is apparent that equation (3.6) yields the derived utility as a continuous function of λ_t, the maximization problem can be passed to optimization routines present in many software packages such as Matlab. Except for possible numerical errors, the formulation (3.9) yields the exact solution for the NT region for a given discretization approach. This algorithm executes in short time even for the large number of nodes in a one-period lattice that we use to approximate a jump-diffusion process, since a limited number of nodes remains inside the NT region at any time τ.

With the use of Lemma 2, we may also state the first order conditions (FOC) for the problem in a closed form.[20]. By substituting from (3.5) into (3.6), differentiating and simplifying, we have the following FOC:

[20] Boyle and Lin (1997) formulated a similar approach for solving a simpler problem but did not reach a closed form solution and so solved the problem for a very small number of periods

$$\sum_{\tau=t+1}^{T-1} \underline{G}_\tau \sum_{i=1}^{\underline{n}_\tau} \Pr\left(\hat{Z}_{\tau,i}\right) \underline{\varphi}(\lambda_t,\tau,i)^\alpha \left[\frac{(1+k)\hat{Z}_{\tau,i}}{\underline{\varphi}(\lambda_t,\tau,i)} - \frac{c(k)}{\lambda c(k)+1}\right]$$
$$+\sum_{\tau=t+1}^{T-1} \bar{G}_\tau \sum_{j=1}^{\bar{n}_\tau} \Pr\left(\hat{Z}_{\tau,j}\right) \bar{\varphi}(\lambda_t,\tau,j)^\alpha \left[\frac{(1-k)\hat{Z}_{\tau,j}}{\bar{\varphi}(\lambda_t,\tau,j)} - \frac{c(k)}{\lambda c(k)+1}\right] \quad (3.7)$$
$$+\sum_{l=1}^{N_T} \Pr\left(\hat{Z}_{T,l}\right) \varphi(\lambda_t,T,l)^\alpha \left[\frac{(1-k)\hat{Z}_{T,l}}{\varphi(\lambda_t,T,l)} - \frac{c(k)}{\lambda c(k)+1}\right]$$

where $\underline{G}_\tau = \alpha V\left(1,\underline{\lambda}_\tau,\tau\right)/(\underline{\lambda}_\tau\left(1+k\right)+1)^\alpha$, $\bar{G}_\tau = \alpha V(1,\bar{\lambda}_\tau,\tau)/(\bar{\lambda}_\tau\left(1-k\right)+1)^\alpha$, $\underline{\varphi}(\lambda_t,\tau,i) = R^{\tau-t} + (1+k)\lambda_t\hat{Z}_{\tau,i}$, $\bar{\varphi}(\lambda,\tau,j) = R^{\tau-t} + (1-k)\lambda_t\hat{Z}_{\tau,j}$, $\varphi(\lambda_t,T,l) = R^{T-t} + (1-k)\lambda_t\hat{Z}_{T,l}$ and $c(k)$ is set equal to $1 + k$ and $1 - k$, respectively for the buy and sell boundary of the NT region. In principle, (3.7) may be solved to yield the boundaries of the NT region instead of brute-force maximization of (3.6); however, this approach doesn't appear to have numerical advantages.

3.2 Numerical estimation

To solve for the NT region for the diffusion case, we approximate the stock dynamics by the Kamrad and Ritchken (1991) trinomial model. For the jump-diffusion case we use a multinomial approximation. When the jump size is lognormally distributed, we approximate the diffusion component by the trinomial model with the mean and variance implied by (2.3) with the probabilities adjusted by the factor $1-\eta\Delta t$. To approximate the jump component, we space $m > 3$ states within the span $\pm 6\sigma_K$ by the same distance in the log scale as in the trinomial model and derive the probabilities as the normalized to 1 densities implied by the distribution of the jump component K. These probabilities are adjusted by the factor $\eta\Delta t$. In the final step, the adjusted trinomial probabilities are added to the adjusted three central probabilities of the jump component. When the jump size is binomially distributed, we search for a trinomial model for the diffusion component such that its span multiplied by a natural number greater than one yields the span of the jump component, a feature that is necessary for the resulting composite lattice to recombine. We then combine both components as before. We lay out further details for our discretization in the appendix.

A straightforward discretization may lead to errors in terms of the mean and volatility of the discretized process relative to similar parameters of the continuous-time process. We solve this problem by adjusting the mean and the variance for the discretized process so that these parameters net of discretization errors exactly match the true ones. As shown in the appendix, these adjusted parameters may be found easily by solving a set of two non-linear equations. It is also shown that the adjusted parameters are close to the true ones.

To derive the NT region, we start at $t = T - \Delta t$ and move recursively backward while using the forward induction (3.1)-(3.2) to solve the problem at each time t. The function (3.9) is derived and passed to an optimization routine, which derives its maximizing arguments. We use a time partition of 250 for one calendar year, which approximately corresponds to daily portfolio revisions.

3.3 The analysis for a frictionless market

As was shown in Merton (1969) and Liu *at al*. (2003) among others, the optimal portfolio policy for a diffusion or jump-diffusion process in continuous time without transaction costs for an isoelastic utility function is myopic. This is not generally true for a *given* time partition and varying horizon length for a discretized process. On the other hand, for any investment horizon length and for a successively finer time partition our discrete-time results clearly tend to their continuous-time counterparts, an expected result under the weak convergence criterion. Here we reproduce the Liu and Loewenstein (2008) result for a mixed process under an exponentially distributed investment horizon. They derive the following function:

$$\rho(\theta) = r + \theta(\mu - r - \eta E[K]) - \theta^2 \sigma^2 \delta / 2 + \eta E[(1+\theta K)^{1-\delta} / (1-\delta)] \tag{3.7}$$

Liu and Loewenstein (2008) showed that the above function is strictly concave in $\theta \leq 1$, and its unique maximizer θ^* yields the optimal risky to riskless asset proportion as $\theta^*/(1 - \theta^*)$.

In our case a program deriving the optimal portfolio proportions for a finite horizon in discrete time results from setting the transaction cost rate k equal to zero in (2.16)-(2.17) and maximizing the terminal utility of wealth in the risky to riskless asset proportion. Note, however, that since at each time period the portfolio is traded to reach this same ratio, which is the sole state variable for the problem, the solution would be the same as for the one-period problem with the given discretization. To take the advantage of the convergence of our discretization to the continuous-time distribution, we successively increase the time partition and solve the frictionless case for the terminal distribution. This approach will allow the verification of the precision of our approach since the continuous-time results are easy to derive.

4 Results

We focus the presentation of our main results on the differences between the diffusion and jump-diffusion cases. Our base case uses the following set of parameters: total volatility σ of 18%, risk premium of 6%, jump intensity η of 0.5, jump volatility σ_K of 7%, transaction costs rate k of 0.5%, relative risk aversion (RRA) δ of 3 and logarithm of the expected jump size μ_K of -2%. In all presented cases the time step, Δt is 1/250.

First, we estimate the results for the frictionless markets ($k = 0$) for the diffusion ($\eta = 0$) and for the mixed process ($\eta = 0.5$). With our discretization the Merton line for the diffusion is 1.6130, compared to the continuous-time optimal risky to riskless asset proportion of 1.6129. Our discrete and continuous-time jump-diffusion counterparts respectively are 1.6081 and 1.6084 indicating good convergence properties of our discretization approach. We see that the optimal investment policy is little different in both cases, which provides an indication for the results in the presence of transaction costs.

[Fig 1 around here]

Figure 1 displays the NT region for the above parameters for $k = 0.5\%$ and for jump intensities η equal to 0 (diffusion) and 2.[21] In Figure 1, we may also clearly observe the convergence of the NT region boundaries to constant levels as the horizon length increases for a given set of parameters.

[Table 1 around here]

Table 1 presents the numerical results for our base case, as well as for variation in all parameters except for the logarithm of the expected jump size μ_K whose changing brought little variation in the results since the diffusion component is compensated for its values to keep the total mean constant. In Panel B, where we increase the total volatility without changing the volatility of the jump component, we observe a convergence of the results for the mixed and pure diffusion processes. In Panels C-E we observe an increasing divergence between the results for the mixed and pure diffusion processes corresponding respectively to increases in the risk premium, jump intensity and jump volatility. From the results in Panel F, there appear to be little effect of varying the transaction cost rate for the relation between the results for the mixed and pure diffusion processes. Last, the results in Panel G for increasing the risk aversion coefficient display a convergence between the mixed and pure diffusion processes. To put the overall differences between the two considered processes in perspective, we focus on the top line of Panel G, where for the RRA of 2 we observe the highest relative divergence between the results in Table 1. This highest divergence for our choice of parameters translates into the difference in the risky asset holdings of the order of 0.1 and 0.2% proportionally to the total wealth, respectively at the buy and sell boundary of the NT region.

[Table 2 around here]

[21] For our base case (η = 0.5), the graphs of the NT region for the diffusion and mixed process almost coincide.

For the parameter values when it is optimal to borrow for the bounded jump size, instead of presenting the boundaries of the NT region in terms of the risky to riskless asset proportions, which are negative in this case, we present more intuitively our results in terms of borrowing per one share of the risky asset. In other words, the investor borrows the indicated quantity to purchase one share of the risky asset, with the remainder of financing coming from her own wealth. It is clear from Table 2 that the borrowing varies significantly with the risk premium while the impact of the jump intensity η is relatively small. In terms of the differences in the risky asset holdings in proportion to the total wealth between the mixed and pure diffusion processes, the bottom line in Panel B, for which these differences are the largest may be translated into the differences of 1.8 and 3.7%, respectively at the buy and sell boundary of the NT region.

From the results in Table 1 it appears, therefore, that for our choice of parameters, when there is no borrowing the optimal policy for the mixed process is very similar to the pure diffusion case as underlined by the highest 0.2% difference for the risky asset holdings in proportion to the total wealth. The differences in the optimal policy between the two cases appear to grow when it is optimal to borrow, as evidenced in Table 2.

An open question of our work is the accuracy of our discrete time numerical algorithm in approximating the continuous time solution in the presence of transaction costs. Unfortunately there are no closed form expressions for the NT region for the jump-diffusion problem (or, for that matter, for simple diffusion) when the investment horizon is finite. For this reason we compare our results to those of a different approximation to the exact solution, by Liu and Lowenstein (2008). Their approximation to the continuous time solution is in the form of an Erlang-distributed horizon that produces a sequence of ordinary differential equations, whose successive solutions converge to the value function and NT region of the fixed-horizon jump-diffusion case.

[Fig 2 around here]

Figure 2 and Table 3 show the NT region for the jump-diffusion case evaluated with our algorithm for the parameter values used by Liu and Lowenstein (LL, 2008). We follow the presentation in that latter study, i.e. we display the reciprocals of the NT region boundaries as defined in this paper with the following set of parameters: the RRA δ of 5, the total volatility σ of12.86%,[22] the risk premium of 7%, the jump intensity η of 0.1, the logarithm of the expected jump size μ_K of -6.75%, the jump volatility σ_K of 8.53%, the transaction cost rate k for stock purchase (sale) of 1% (0).

[Table 3 around here]

Although the exact numerical values of LL are not available, it is clear that our diagram is virtually identical to the corresponding LL Figure 6. Hence, there is reason to believe that our numerical algorithm provides equally good convergence and approximation properties to the "true" continuous time solution as the alternative approximation through the Erlang-distributed horizon of the Liu-Lowenstein (2002, 2008) approach.

[22] This value of the total volatility corresponds to the value of the diffusion component of 12.39%.

5 Concluding Remarks

We presented an efficient numerical solution to the problem of deriving the NT region in a discrete-time finite-horizon case for iid risky asset returns. The solution to our main research question indicates that the major factor driving the NT region for the mixed process apart from its diffusion counterpart is the jump intensity. It remains an empirical question whether the jump intensity estimated from market data would lead to major changes in portfolio rules compared to the simple diffusion case. Further, an empirical study may examine relative gains or losses in derived utility resulting from the adoption of either investment policy and tested with the observed paths of an index.

A factor that may modify the influence of the jump intensity on the NT region is intermediate consumption. We hypothesize that with intermediate consumption even relatively low jump intensity may lead to portfolio rules for the mixed process relatively far apart from its diffusion counterpart. The reason for this conjecture is the plausibility that the risk aversion will sway an agent from holding the risky asset given that a large proportion of the risky asset in the agent's portfolio may lead to low consumption states at intermediate dates in the presence of jumps. The verification of this conjecture should be a topic of future research.

Appendix

Proof of Lemma 1[23]

The characteristic function of the terminal stock price at time T for a \$1 initial price under the jump-diffusion process (2.3) is

$$\begin{aligned}\varphi_{JD}(\omega) &= \exp(i\omega\mu_d T - \frac{\omega^2\sigma_d^2 T}{2})\exp(-\eta T)\sum_{N=0}^{\infty}\frac{(\eta T)^N}{N!}[\varphi_J(\omega)]^N \\ &= \exp(i\omega\mu_d T - \frac{\omega^2\sigma_d^2 T}{2})\exp[\eta T(\varphi_J(\omega)-1)],\end{aligned} \tag{A.1}$$

where $\varphi_J(\omega)$ is the characteristic function of the jump distribution. The first exponential corresponds to the diffusion component and the second to the jump component.

The characteristic function of the discretization (2.7) is

$$\varphi(\omega) = \eta\Delta t\varphi_J(\omega) + (1-\eta\Delta t)[\exp(i\omega\mu_d\Delta t)\varphi_\varepsilon(\omega\sigma_d\sqrt{\Delta t})], \tag{A.2}$$

where $\varphi_\varepsilon(\omega)$ is the characteristic function of ε.[24] Since the distribution of ε has mean 0 and variance 1, we have

$$\begin{aligned}&E[\varepsilon] = 0 = i\varphi_\varepsilon'(0) \\ &E[\varepsilon^2] = 1 = -\varphi_\varepsilon''(0)\end{aligned}$$

By the Taylor expansion of $\varphi_\varepsilon(\omega)$, we get

$$\varphi(\omega) = \eta\Delta t\varphi_J(\omega) + (1-\eta\Delta t)\left[\exp(i\omega\mu_d\Delta t)[1-\frac{\omega^2\sigma_d^2\Delta t}{2} + \omega^2\sigma_d^2\Delta t h(\omega\sigma_d\sqrt{\Delta t})]\right],$$

---

[23] The proof is similar to that of Theorem 21.1 in Jacod and Protter (2003).

[24] If, instead of (2.2) we have a convolution of the diffusion and jump components then the characteristic function becomes $\varphi(\omega) = (\eta\Delta t\varphi_J(\omega) + 1 - \eta\Delta t)[\exp(i\omega\mu_d\Delta t)\varphi_\varepsilon(\omega\sigma_d\sqrt{\Delta t})]$. The multiperiod convolution, however, still converges to (A.3).

where $h(\omega) \to 0$ as $\omega \to 0$. The multiperiod convolution has the characteristic function $\varphi(\omega)^{T/\Delta t}$. Taking the limit, we have

$$
\begin{aligned}
&\lim_{\Delta t \to 0} [\varphi(\omega)]^{T/\Delta t} = \\
&\lim_{\Delta t \to 0} \exp\left[\frac{T}{\Delta t} \left(\ln(\eta \Delta t \varphi_J(\omega) + (1-\eta\Delta t) \left[\exp(i\omega\mu_d \Delta t)[1 - \frac{\omega^2 \sigma_d^2 \Delta t}{2} + \omega^2 \sigma_d^2 \Delta t h(\omega\sigma_d \sqrt{\Delta t})] \right] \right) \right] \\
&= \exp\left[\eta T \left(\varphi_J(\omega) - 1 \right) + i\omega\mu_d T - \frac{\omega^2 \sigma_d^2 T}{2} \right]
\end{aligned}
\tag{A.3}
$$

after applying l'Hospital's rule. (A.3) is, however, the same as (A.1), the characteristic function of (2.3), and Levy's continuity theorem[25] proves the weak convergence of (2.4) to (2.3), QED.

Proof of Lemma 2

The lemma may be demonstrated very simply by induction. Without loss of generality, in our proof we consider only the portfolio adjustments to the lower boundary of the NT region $\underline{\lambda}_\tau$ and the resulting quantity $\underline{X}_\tau$. The inclusion of the adjustment to the other boundary $\overline{\lambda}_\tau$ follows easily by extension. Consider $t = T-2$. We have $V\left(1, \underline{\lambda}_{T-1}, T-1\right) = \alpha^{-1} \sum_{s=1}^{m} p_s \left(R + (1-k)\underline{\lambda}_{T-1} z_s\right)^\alpha$, where p_i is the probability of a one-period stock return z_i, $s = 1...m$. With the use of the lemma, we have:

$$
\Delta J_{T-1} = \alpha^{-1} \sum_{i=1}^{\underline{n}_{T-1}} p_i \left(\frac{R + (1+k)\lambda_{T-2} z_i}{\underline{\lambda}_{T-1}(1+k) + 1} \right)^\alpha \sum_{s=1}^{m} p_s \left(R + (1+k)\underline{\lambda}_{T-1} z_s\right)^\alpha, \tag{A.4}
$$

where the first summation results from the definition of $\underline{X}_\tau$, $\tau = T-1$, and $m \geq \underline{n}_{T-1} \geq 0$. By the homothetic property which in this simple case collapses to multiplying terms under the same power, (3.6) yields the same result we would get from equation (3.2) by considering each path separately.

[25] See for instance Jacod and Protter (2003), Theorem 19.1.

Consider any time *t*. Assume that the lemma holds at $\tau+1$. At time τ we have:

$$\Delta J_{\tau} = \sum_{i=1}^{n_{\tau}} \Pr\left(\hat{Z}_{\tau,i}\right) V\left(\frac{R+(1+k)\lambda_i \hat{Z}_i}{\underline{\lambda}_{\tau}(1+k)+1}, \frac{R+(1+k)\lambda_i \hat{Z}_i}{\underline{\lambda}_{\tau}(1+k)+1}\underline{\lambda}_{\tau}, \tau\right) \equiv \sum_{i=1}^{n_{\tau}} \underline{X}_{\tau} V\left(1, \underline{\lambda}_{\tau}, \tau\right). \quad \text{(A.5)}$$

When we apply one forward inductive step to the quantity above we get:

$$\Delta J_{\tau} = \sum_{i=1}^{n_{\tau}} \underline{X}_{\tau} \left[\begin{array}{l} \sum_{k=1}^{\underline{n}_{\tau+1}} p_k \left(\frac{R+(1+k)\underline{\lambda}_{\tau} z_k}{\underline{\lambda}_{\tau+1}(1+k)+1}\right)^{\alpha} V\left(1, \underline{\lambda}_{\tau+1}, \tau+1\right) + \\ \sum_{j=1}^{\overline{n}_{\tau+1}} p_j \left(\frac{R+(1-k)\underline{\lambda}_{\tau} z_j}{\overline{\lambda}_{\tau+1}(1-k)+1}\right)^{\alpha} V\left(1, \overline{\lambda}_{\tau+1}, \tau+1\right) + \\ \sum_{s=1}^{m-\underline{n}_{\tau+1}-\overline{n}_{\tau+1}} p_s V\left(R, \underline{\lambda}_{\tau} z_s, \tau+1\right) \end{array}\right], \quad \text{(A.6)}$$

where the first two summations in square brackets consider these one-period paths that are outside the NT region by using the induction hypothesis, and the third one considers these paths which remain inside this region, with z_k's, z_j's and z_s's denoting appropriate one-period returns and *m* denoting the number of one-period returns in a given lattice. Since, by the definition of $\underline{X}_{\tau}$ it is apparent that equation (A.4) considers all the relevant path information, continuing forward until the terminal date *T* is reached will reproduce the definition (3.2) for the time-τ paths outside the NT region. This ends the proof, QED.

Lattice Construction

In this section, we explain the setup for our lattice and the way we control for the ensuing discretization errors. As the first building block of the lattice representing the diffusion component, we use the Kamrad-Ritchken (1991) trinomial model. This model implies a certain spacing for the jump component when we approximate for a lognormally distributed jump while we truncate the one-period distribution of jumps to $\pm 6\sigma_K$. When the jump component is binomially distributed, the binomial spacing will determine the spacing for the diffusion component. Before setting out with estimating the NT region, we modify the trinomial model as explained below to insure the convergence with respect to the first and second moments of the distribution as the time partition increases. This is an important aspect of our work since the trinomial model converges only approximately to the true parameters, and relatively small differences in the mean and variance may yield significant differences in the estimates for the optimal investment policy. To streamline the presentation, we first present the construction of our lattice for the pure diffusion case, with the jump-diffusion case following easily by extension.

To insure the convergence to the true parameters μ and σ, we solve a set of two non-linear equations for the 'nominal' values μ_n and σ_n, which now are apparently equal to the respective 'nominal' diffusion parameters, μ_d and σ_d:

$$\begin{aligned} &\log\left(P^{'}\exp(H)\right) - \mu T = 0 \\ &\text{and} \\ &P^{'}H^{2} - (P^{'}H)^{2} - \sigma^{2}T = 0 \end{aligned} \quad , \tag{A.7}$$

where P and H respectively are the vectors resulting from the N-period convolutions of the trinomial single-period probabilities p and log-returns h with themselves as stated below, the square of a vector is element-by-element, and $N = T/\Delta t$. For a single period with the time step Δt we have the following:

$$\begin{aligned} &p_{1,3} = 1/2\kappa^{2} \mp (\mu_d - \sigma_d^{2}/2)\sqrt{\Delta t}/2\sigma_d,\ p_2 = 1 - 1/\kappa^{2} \\ &\text{and} \\ &\Delta h = \kappa\sigma_d\sqrt{\Delta t},\ h_{1,3} = \mp\Delta h,\ h_2 = 0 \end{aligned} \quad , \tag{A.8}$$

where $\kappa \geq 1$ is the Kamrad-Ritchken (1991) stretch parameter. Note that from the formulation (A.7)-(A.8) it is apparent that solving for the nominal values μ_d and σ_d will in turn determine the parameters for the trinomial model.

For a mixed jump-diffusion process with lognormal jumps, to construct our now multinomial lattice by assuring the convergence to the true total mean μ and volatility σ we change the parameters of the trinomial distribution by setting $\mu_d = \mu_n - \eta\mu_K$ and $\sigma_d^2 = \sigma_n^2 - \eta(\sigma_K^2 + \mu_K^2)$ in (A.8) with μ_n and σ_n now representing the total nominal mean and volatility of the process. Having set the approximation for the diffusive component, to approximate for a lognormally distributed jump we set the equally spaced log-states g

from $-n_K\Delta h$ to $n_K\Delta h$, with n_K being the smallest integer which assures that those states span at least $\pm 6\sigma_K$.[26] This setting results in a multinomial recombining lattice with a total number of one-period states $2n_K + 1$. The probabilities of the jump component states are then taken as normalized to 1 normal densities $\phi(i\Delta h - \mu_K + \sigma_K^2/2)$, $i = -n_K \ldots n_K$. To arrive at the final single-period probability vector π, we weigh these normalized normal densities by $\eta\Delta t$ and add to the three central ones the trinomial probabilities weighted by $1-\eta\Delta t$. With the vectors P and H in (A.7) now respectively representing the N-period convolutions with themselves of the multinomial one-period probabilities π and log-states g, we solve the system (A.7) as before. Unfortunately, for this type of multinomial approximation we apply, which is rather crude for the jump component, we arrive at the nominal values μ_n and σ_n rather far apart from the true values μ and σ. This apparently poses a problem of whether we discretize the right process. We solve this problem by optimizing over the stretch parameter κ an objective function of the form $(\mu - \mu_n)^2 + (\sigma - \sigma_n)^2$, which yields excellent results as we demonstrate below.

In our last case, where we consider a discrete, binomially distributed jump size the procedure is slightly different. We first set the binomial probabilities for the jump component, which is made simply by substituting μ_K and σ_K respectively for μ_d and σ_d and setting $\kappa = 1$ in (A.8), now with the log-step ($\equiv\Delta k$) set equal to $\sigma_K\sqrt{\Delta t}$. Now the crux of constructing a recombining multinomial lattice is to find a trinomial lattice for the diffusion component whose step Δh satisfies $\Delta k/\Delta h = n$, where n is a natural number greater than 1. This may be done very simply by setting n equal to the smallest natural number exceeding $\Delta k/\sigma_K\sqrt{\Delta t}$. This, in turn yields the stretch parameter $\kappa = \Delta k / n\sigma_d\sqrt{\Delta t}$ for the trinomial model representing the diffusive component. In the final step we proceed analogously with the lognormally distributed jump size above to adjust the lattice to yield the true total mean μ and volatility σ for multi-period convolutions. Notice that in this last case the stretch parameter κ is fixed by the jump

[26] For a sufficiently large time step Δt we may have the log-states for the jump component approximation lie inside the log-states for the diffusive component approximation. This is not the case for the parameter choices in this work, even for the largest considered Δt of 1/50.

component; however, this fact does not cause the nominal values μ_n and σ_n to lie far apart from the true values μ and σ.

Last, we present the errors for applying a non-adjusted lattice. In all cases, we set $\mu = 0.1$ $\sigma = 0.18$, $\Delta t = 1/250$ and T = 1. The jump intensity η was set to 0.5. For a pure diffusion, we recovered $\mu = 0.0999$ and $\sigma = 0.1798$; for lognormally distributed jump size respectively 0.9998 and 0.1871; and for a binomially distributed jump size respectively 0.9994 and 0.1792. Note that even for a pure diffusion case, where errors are relatively small, they still produce 'visible' errors on the estimates for the NT region. The nominal values μ_n and σ_n for the three considered cases in their respective order as above were as follows: 0.1000 and 0.1802; 0.1000 and 0.1801; and 0.1001 and 0.1801.

References

Boyle, P. P. and X. Lin, 1997, “Optimal Portfolio Selection with Transaction Costs,” *North American Actuarial Journal* 1, 27-39.

Carr, P., 1998, “Randomization and the American Put,” *Review of Financial Studies* 11, 597-626.

Constantinides, G. M., 1979, “Multiperiod Consumption and Investment Behavior with Convex Transactions Costs,” *Management Science* 25, 1127-37.

Constantinides, G. M., 1986, “Capital Market Equilibrium with Transaction Costs,” *Journal of Political Economy* 94, 842-862.

Davis, M. H. A. and A. R. Norman, 1990, “Portfolio Selection with Transaction Costs,” *Mathematics of Operations Research* 15, 676-713.

Dumas, B. and E. Luciano, 1991, “An Exact Solution to a Dynamic Portfolio Choice Problem under Transactions Costs,” *Journal of Finance* 46, 577-596.

Genotte, G. and A. Jung, 1994, “Investment Strategies under Transaction Costs: The Finite Horizon Case,” *Management Science* 40, 385-404.

Kamrad, B. and P. Ritchken, 1991, “Multinomial Approximating Models for Options with k State Variables,” *Management Science* 37, 1640-1652.

Jacod, J. and P. E. Protter, *Probability Essentials*, Springer Verlag, 2003.

Liu, H. and M. Loewenstein, 2002, “Optimal Portfolio Selection with Transaction Costs and Finite Horizons,” *Review of Financial Studies* 15, 805-835.

Liu, H. and M. Loewenstein, 2007, “Optimal Portfolio Selection with Transaction Costs and ‘Event Risk’,” working paper, http://ssrn.com/abstract=965263.

Liu, J., F., A. Longstaff and J. Pan, 2003, “Dynamic Asset Allocation with Event Risk,” *Journal of Finance* 58, 231-259.

Magill, M. J. P., and G. M. Constantinides, 1976, “Portfolio Selection with Transactions Costs,” *Journal of Economic Theory* 13, 245-263.

Merton, R. C., 1969, “Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case,” *Review of Economics and Statistics* 51, 247-257.

Merton, R. C., 1971, "Optimum Consumption and Portfolio Rules in a Continuous-Time Model," *Journal of Economic Theory* 3, 373-413.

Oancea, I. M. and S. Perrakis, 2009, "Jump-Diffusion Option Valuation Without a Representative Investor: A Stochastic Dominance Approach", working paper, Concordia University, http://ssrn.com/abstract=1360800.

Table 1
Sensitivity of the Boundaries of the NT Region

Variable Parameter	$\underline{\lambda}_{jd}$	$\overline{\lambda}_{jd}$	$1-\underline{\lambda}_{jd}/\underline{\lambda}_d$ (%)	$1-\overline{\lambda}_{jd}/\overline{\lambda}_d$ (%)
		A: Base Case		
None	1.3020	2.0017	0.27	0.59
σ		B: Sensitivity to Total Volatility		
16%	2.7911	4.5080	1.70	2.31
20%	0.8091	1.2336	0.07	0.17
22%	0.5667	0.8679	0.01	0.04
$\mu - r$		C: Sensitivity to Risk Premium		
4%	0.562	0.863	0.06	0.05
5%	0.857	1.308	0.04	0.25
7%	2.052	3.231	0.81	1.20
η		D: Sensitivity to Jump Intensity		
0.25	1.304	2.008	0.13	0.29
1	1.299	1.990	0.52	1.17
2	1.292	1.967	1.01	2.31
σ_K		E: Sensitivity to Jump Volatility		
6%	1.3028	2.0054	0.20	0.41
8%	1.3010	1.9972	0.34	0.81
9%	1.3000	1.9915	0.42	1.10
k		F: Sensitivity to Transaction Costs Rate		
0.1%	1.425	1.816	0.26	0.55
1%	1.226	2.128	0.28	0.61
2%	1.117	2.270	0.28	0.62
RRA		G: Sensitivity to Risk Aversion		
2	8.546	17.997	2.49	3.68
5	0.488	0.701	0.03	0.31
10	0.190	0.266	0.04	0.21

The table displays the results for the NT region for a 10-year investment horizon. Other parameters for the base case are as follows: the transaction costs rate k is 0.5%, the RRA δ is 3, the total volatility σ is 18%, the risk premium $\mu - r$ is 6%, the logarithm of the expected jump size μ_K is -2%, the jump volatility σ_K is 7%, the jump intensity η is 0.5. In Panels B-G the name and values for a parameter different than in the base case are provided in the first column.

Table 2
Borrowing per One Share of Risky Asset

η	$\left\lvert\underline{\lambda}_{jd}^{-1}\right\rvert$ (%)	$\left\lvert\overline{\lambda}_{jd}^{-1}\right\rvert$ (%)	$\left\lvert\underline{\lambda}_{d}^{-1}\right\rvert-\left\lvert\underline{\lambda}_{jd}^{-1}\right\rvert$ (%)	$\left\lvert\overline{\lambda}_{d}^{-1}\right\rvert-\left\lvert\overline{\lambda}_{jd}^{-1}\right\rvert$ (%)
A: Risk Premium 7%				
0.5	4.61	9.43	0.09	0.14
2	4.37	9.01	0.33	0.56
B: Risk Premium 8%				
0.5	14.34	22.72	0.93	1.70
2	14.06	22.22	1.31	2.20

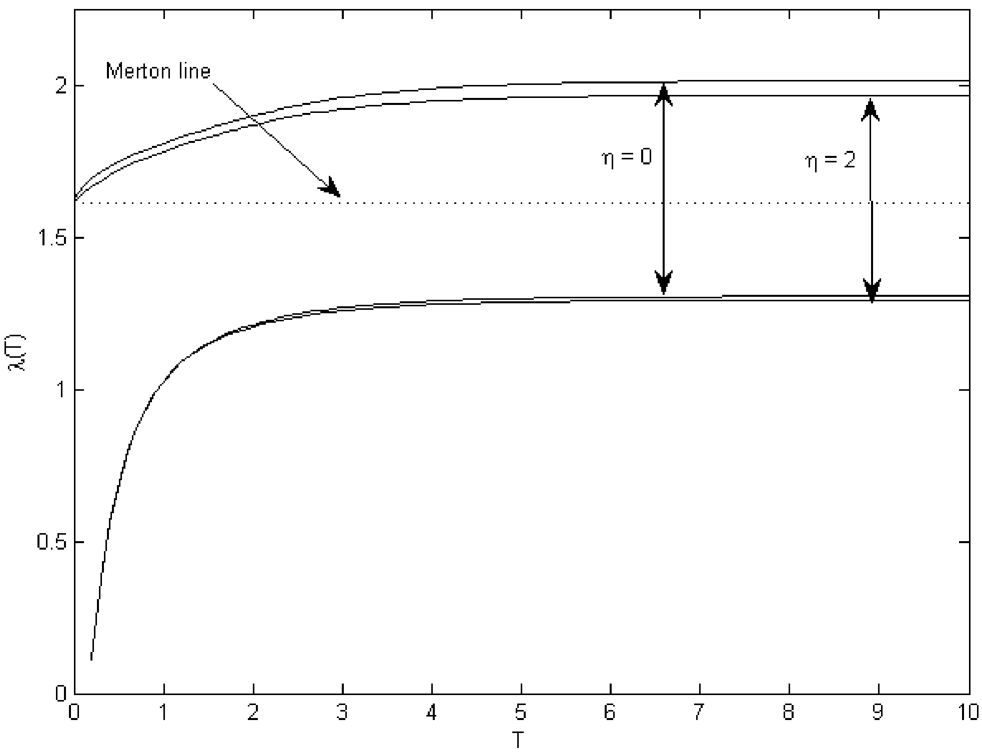
The table displays the results for the NT region for a 10-year investment horizon in terms of the borrowing to finance one share of the risky asset. The parameters are as follows: the transaction costs rate k is 0.5%, the RRA δ is 2, the total volatility σ is 18%, the riskless rate r is 4%, the logarithm of the expected jump size μ_K is -2%, the jump volatility σ_K is 7%.

Table 3
Results for Liu-Loewenstein (2008) Parameters

T	$1/\overline{\lambda}_{jd}$	$1/\underline{\lambda}_{jd}$
0.25	0.194	1.760
0.5	0.190	0.665
1	0.185	0.397
2	0.179	0.299
5	0.167	0.256
10	0.161	0.247
15	0.161	0.246
20	0.161	0.246
25	0.161	0.246

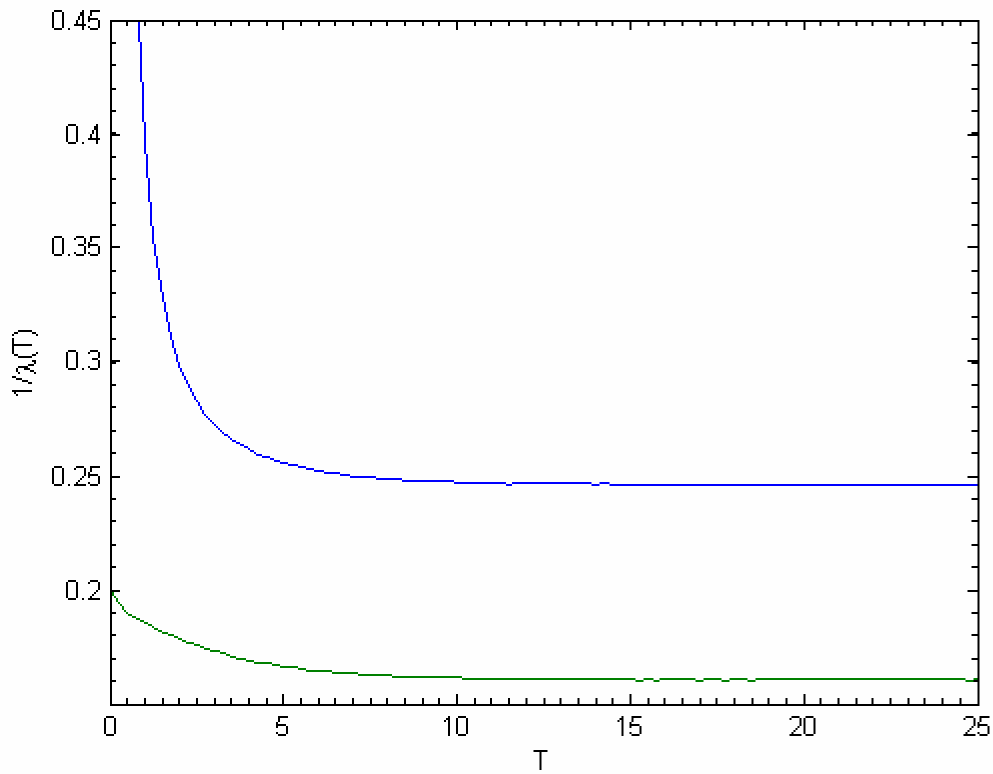
The table displays the results for the NT region for mixed jump-diffusion process for the Liu-Loewenstein (2008) parameters: the RRA δ = 5, the total volatility σ = 12.86%, the risk premium of 7%, the jump intensity η = 0.1, the logarithm of the expected jump size μ_K = -6.75%, the jump volatility σ_K = 8.53%, the transaction cost rate for stock purchase (sale) of 1% (0).

Figure 1: No Transaction Region for Diffusion and Mixed Process



The figure displays the results for the NT region as a function of the investment horizon T for mixed jump-diffusion process (η = 2) and for diffusion process (η = 0). The other parameters are as follows: the total volatility σ of 18%, risk premium of 6%, jump intensity η of 0.5, jump volatility σ_K of 7%, transaction costs rate k of 0.5%, relative risk aversion (RRA) δ of 3 and logarithm of the expected jump size μ_K of -2%.

Figure 2: No Transaction Region for Liu-Loewenstein (2008) Parameters



The figure displays the results for the NT region as a function of the investment horizon T for mixed jump-diffusion process for the Liu-Loewenstein (2008) parameters: the total volatility σ of 12.86%, risk premium of 7%, jump intensity η of 0.1, jump volatility σ_K of 8.53%, transaction cost rate for stock purchase (sale) of 1% (0), relative risk aversion (RRA) δ of 5 and logarithm of the expected jump size μ_K of -6.75%.