Problem 1:  (a) Prove that if \(a + b = c\), then \(\max\{s_1a, s_2b\} \geq \frac{1}{\frac{1}{s_1} + \frac{1}{s_2}}c\), for any \(a, b \geq 0, s_1, s_2 > 1\).

(b) We define the harmonic mean \(H\) and the geometric mean \(G\) as follows:
\[
H(a, b) = \frac{2}{\frac{1}{a} + \frac{1}{b}}, \quad G(a, b) = \sqrt{a \cdot b}, \quad a, b > 0.
\]
Prove that \(H \leq G\).

(c) Let \(H, G\) be as above. Let \(a_0 = 1, b_0 = 2\) and \(a_{n+1} = H(a_n, b_n), b_{n+1} = G(a_n, b_n), n = 0, 1, 2, \ldots\). Prove that both sequences \(\{a_n\}_{n \geq 0}\) and \(\{b_n\}_{n \geq 0}\) are convergent and have the same limit. Can you find the limit?

Problem 2:  (a) Let \(I\) denote the identity matrix. For square matrices \(A, B\), prove that if the matrix \(I - AB\) is invertible, then the matrix \(I - BA\) is also invertible.

(b) Let \(S\) and \(T\) be linear subspaces of \(\mathbb{R}^n\). Prove: If
\[
\dim(\text{Span}(S \cup T)) = \dim(S \cap T) + 1,
\]
then one of the subspaces is a subset of the other. \(\text{Span}(W)\) is the smallest linear subspace containing set \(W\).

(c) Let \(\mathcal{M}^n\) denote the set of all \(n \times n\) real matrices. Let \(A \in \mathcal{M}^n\) be of rank \(0 \leq k \leq n\). Let \(\mathcal{L} = \{B \in \mathcal{M}^n : BA = 0\}\) and \(\mathcal{R} = \{C \in \mathcal{M}^n : AC = 0\}\). Show that \(\mathcal{L}\) and \(\mathcal{R}\) are linear spaces and compute their dimensions.

(d) Compute the value of the determinant of the \(3 \times 3\) complex matrix \(A\), provided that \(\text{tr}(A) = 1, \text{tr}(A^2) = -3, \text{tr}(A^3) = 4\). [Here \(\text{tr}(A)\) denotes the the trace, that is, the sum of the diagonal entries of the matrix \(A\).]
Problem 3: Let $f: X \to Y$ be a function between two metric spaces. Are the following statements true or false? Give a sketch of a proof or a counterexample.

(a) If $f$ is continuous and $f(K)$ is complete in $Y$, then $K$ is complete in $X$.
(b) If $f$ is continuous and $K$ is complete in $X$, then $f(K)$ is complete in $Y$.
(c) If $X$ is compact and $f$ is continuous, then for any open $U \subset X$ the image $f(U) \subset Y$ is also open.
(d) If $f$ is continuous and $f(G) \subset Y$ is dense in $Y$, then $(G) \subset X$ is also dense in $X$.

Problem 4:

(a) Prove: If both $f = u + iv$ is analytic in an open neighbourhood of $z_0$ and $f'(z_0) \neq 0$, then the lines $u = \text{const}$ and $v = \text{const}$ are perpendicular at $z_0$.

(b) (i) For $A > 0$ show
\[ \int_0^{\pi} \exp(-A \sin t)dt < \frac{\pi}{A}. \]
Hint: Show first that $t \in (0, \pi/2)$ implies $\sin t > 2t/\pi$.
(ii) Prove
\[ \int_0^{+\infty} \frac{\cos x}{x^2 + a^2}dx = \frac{\pi}{2a}e^{-a}, \quad a > 0. \]

Problem 5:

Let us define the space $X = \{z \in \mathbb{C}: |z| \geq 1\}$ and a function $d(z_1, z_2)$ as follows:

(i) if $\arg z_1 = \arg z_2$, then $d(z_1, z_2) = |z_1 - z_2|$;

(ii) if $\arg z_1 \neq \arg z_2$, then
\[ d(z_1, z_2) = |z_1| - 1 + \left(\text{length of the shorter arc between } \frac{z_1}{|z_1|} \text{ and } \frac{z_2}{|z_2|}\right) + |z_2| - 1. \]

(a) Show that $d$ defines metric on $X$ (for triangle inequality you can consider just one case of possible positioning of $z_1, z_2, z_3$);

(b) Sketch the balls: $B(3, 1), B(i, 1), B(3/2, 1)$;

(c) Is $(X, d)$ a complete metric space?

(d) Is $(X, d)$ a compact metric space?

(e) Is $(X, d)$ a separable metric space?

(f) Let $\rho$ be the standard Euclidean metric on $X$. Is the identity a continuous map from $(X, \rho)$ onto $(X, d)$?
Problem 6: Consider the measure space \( \{\mathbb{R}, \mathcal{L}, m\} \), where \( m \) is Lebesgue measure.

(a) State Fatou’s lemma.

(b) Let \( f \in L^1(\mathbb{R}, \mathcal{L}, m) \). Prove that
\[
\forall \epsilon > 0 \; \exists \; \delta > 0 \; \forall \; B \in \mathcal{L} \; m(B) < \delta \Rightarrow \int_B |f| \, dm < \epsilon ,
\]
i.e., Lebesgue integral of \( f \) is absolutely continuous with respect to Lebesgue measure \( m \).

(c) Assume that \( f_n, g_n \in L^1(\mathbb{R}, m) \), \( f_n \to 0 \) and \( g_n \to 0 \) almost everywhere, as \( n \to +\infty \).

Prove
\[
\lim_{n \to +\infty} \int_A \frac{2f_n(x)g_n(x)}{1 + f_n^2(x) + g_n^2(x)} \, dm(x) = 0 ,
\]
for any set \( A \subset \mathbb{R} \) of finite measure. Show by an example that this does not extend to the whole \( \mathbb{R} \).