Problem 1: Consider the measure space \( \{ \mathbb{R}, \mathcal{L}, m \} \), where \( m \) is Lebesgue measure.

(a) State Fatou’s lemma.

(b) Let \( \{ f_n \} \) be a sequence of integrable functions satisfying \( 0 \leq f_{n+1} \leq f_n \) almost everywhere for all \( n \geq 1 \). Prove: If \( \int_{\mathbb{R}} f_n \, dm \to 0 \), as \( n \to \infty \), then \( f = \lim_{n \to \infty} f_n \) exists and is equal to 0 almost everywhere.

(c) Prove that \( \int_{\mathbb{R}} |f| \, dm = 0 \) implies \( f = 0 \) almost everywhere.

(d) Prove
\[
\lim_{n \to +\infty} \int_{0}^{n} \left( 1 + \frac{x}{n} \right)^n e^{-2x} \, dm(x) = 1 .
\]

Problem 2: (a) A matrix \( A \) is said to be skew-symmetric if \( A^T = -A \) (\( A^T \) is the transposition of \( A \)). Prove: If matrix \( A \) is skew symmetric, then \( A^2 \) is symmetric and for any vector \( x \) we have \( x^T A^2 x \leq 0 \).

(b) (b1) Is there a linear transformation from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) which is represented by a diagonal matrix when written with respect to any fixed basis?

(b2) True or false: A linear transformation \( T : V \to V \) that is onto, must be an isomorphism.

(b3) True or false: if \( A \) is a 3 \( \times \) 3 matrix that has three different eigenvalues, then \( A \) is diagonalizable.

(b4) True or false: if the set \( \{ u, v, w \} \) is linearly independent, then so is
\[
\{ u, u - v, u + v + w \} .
\]

(b5) i) True or false: Every \( n \times n \) complex matrix has an eigenvalue.

(b5) ii) True or false: Every \( n \times n \) real matrix has an eigenvalue.
An $n \times n$ circulant matrix $C$ takes the form

$$C = \begin{bmatrix}
  c_0 & c_{n-1} & \cdots & c_2 & c_1 \\
  c_1 & c_0 & c_{n-1} & \cdots & c_2 \\
  \vdots & c_1 & c_0 & \ddots & \vdots \\
  c_{n-2} & \vdots & \ddots & \ddots & c_{n-1} \\
  c_{n-1} & c_{n-2} & \cdots & c_1 & c_0
\end{bmatrix}.$$ 

Let $\omega_j = \exp(2\pi ij/n)$, $j = 0, 1, \ldots, n - 1$, be the $n$-th roots of unity. Show that the column vectors

$$v_j = (1, \omega_j, \omega_j^2, \ldots, \omega_j^{n-1})$$ 

$j = 0, 1, \ldots, n - 1$, are the eigenvectors of $C$ and that they are linearly independent.

**Problem 3 :** Let $f : X \to Y$ be a function between two metric spaces.

(a) Prove that $f$ is continuous if and only if for any $B \subset Y$

$$f^{-1}(\text{Int}(B)) \subseteq \text{Int}(f^{-1}(B)),$$

where $\text{Int}(A)$ denotes the interior of $A$.

(b) Is the following true: If $f$ is continuous and $K$ is complete in $X$, then $f(K)$ is complete in $Y$. (Justify your answer.)

(c) Is the following true: If $X$ is compact and $f$ is continuous, then for any closed $K \subset X$ the image $f(K) \subset Y$ is also closed. (Justify your answer.)

**Problem 4 :** (a) Let $x_0 = 2$ and $x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n}$ for $n \geq 0$. Prove that the sequence $\{x_n\}_{n \geq 0}$ is convergent and find its limit.

What happens if we change $x_0$ to $x_0 = 2012$?

(b) Consider the functional series

$$F(t) = \sum_{n=1}^{\infty} \frac{1}{n} \sin^n(\pi t), \quad t \in [0, 1].$$

Is $F$ well defined on $[0, 1]$?

Show that $F$ is continuous on $[0, a]$ for any $a < 1/2$.

Is $F$ differentiable on $[0, a]$ for any $a < 1/2$? (Justify your answer.)

(c) Function $f$ is continuous on a circle. Show that there exists a diameter such that the values of $f$ are equal on the diameter ends.
Problem 5:
(a) Prove: If \( f = u + iv \) is analytic in an open domain \( D \) and \( |f| \) is constant in \( D \), then \( f \) is constant.

(b) Does there exist a function \( f \) analytic in the plane satisfying \( f(1/n) = 1/n^2 \) for all \( n \geq 1 \) and \( f(5) = 5 \)? (Justify your answer.)

(c) How many zeros has \( f(z) = z^5 + 3z^3 + 7 \) in the disk \( D(0, 2) \)? How many zeros has \( f(z) = z^5 + 3z^3 + 7 \) in the annulus between the disks \( D(0, 1) \) and \( D(0, 2) \)?

(d) Evaluate:
\[
\int_{-\infty}^{+\infty} \frac{\cos x}{x^2 + 16} \, dx.
\]

Problem 6:
Let us define the space \( X = \mathbb{R}^2 \) and a function \( d((x_1, y_1), (x_2, y_2)) \) as follows:
(i) if \( x_1 = x_2 \), then \( d((x_1, y_1), (x_2, y_2)) = |y_1 - y_2| \);
(ii) otherwise \( d((x_1, y_1), (x_2, y_2)) = |x_1 - y_1| + \sqrt{2} |x_1 - x_2| + |x_2 - y_2| \).

(a) Show that \( d \) defines metric on \( X \) (for triangle inequality you can consider just one case of possible positioning of points \( (x_1, y_1), (x_2, y_2), (x_3, y_3) \));

(b) Sketch the balls: \( B((3, 0), 1), B((1, 1), 1) \);

(c) Is \( (X, d) \) a complete metric space? (Justify your answer.)

(d) Is \( (X, d) \) a compact metric space? (Justify your answer.)

(e) Is \( (X, d) \) a separable metric space? (Justify your answer.)

(f) Let \( \rho \) be the standard Euclidean metric on \( X \). Is the identity a continuous map from \( (X, \rho) \) onto \( (X, d) \)? Is the identity a continuous map from \( (X, d) \) onto \( (X, \rho) \)? (Justify your answer.)