Problem 1: (a) Let 
\[ f(x) = a_1 x^{n_1} + a_2 x^{n_2} + a_3 x^{n_3} + a_4 x^{n_4}, \]
for non-zero \( a_1, a_2, a_3, a_4 \) and pairwise different non-negative integers \( n_1, n_2, n_3, n_4 \). Show that \( f \) has at most 3 zeros in the open interval \((0, +\infty)\).

(b) Consider the series
\[ S(a) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(1+a)^n}, \quad a \geq 0. \]

(i) Prove that \( S(a) \) converges absolutely for all \( a > 0 \) and conditionally for \( a = 0 \).

(ii) Prove that \( S(a) \) converges uniformly on \([0, +\infty)\).

(iii) Find explicitly \( S(a) \).

Problem 2: A matrix \( A = (a_{ij})_{1 \leq i, j \leq n} \) is called skew-symmetric if \( a_{ij} = -a_{ji} \) for all \( 1 \leq i, j \leq n \).

(a) Show that if \( A \) is skew-symmetric and \( n \) is odd, then \( \det A = 0 \);

(b) Show that if \( A \) is skew-symmetric and \( n \) is even, then \( \det A = \det B \), where \( b_{ij} = c + a_{ij} \), for all \( 1 \leq i, j \leq n \) and \( c \) is a constant.

(c) Let
\[ A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & i \end{bmatrix}. \]
Find an unitary matrix \( U \) and a diagonal matrix \( D \) such that \( U^{-1}AU = D \). (Here \( i = \sqrt{-1} \).)
Problem 3: Let $f : X \to Y$ be a function between two metric spaces. Are the following statements true or false? Give a sketch of a proof or a counterexample.

(a) If for any open set $U \subset X$ the set $f(U) \subset Y$ is also open, then $f$ is continuous.
(b) If $f$ is continuous and $U \subset Y$ is open, then $f^{-1}(U) \subset X$ is also open.
(c) If $X$ is compact and $f$ is continuous, then for any closed $F \subset X$ the image $f(F) \subset Y$ is also closed.
(d) If $f$ is continuous and $G \subset X$ is nowhere dense, then $f(G) \subset Y$ is also nowhere dense.

Problem 4:
(a) Prove: If both $f$ and $\bar{f}$ are analytic in an open connected region $\Omega$, then they are constant in $\Omega$.
(b) Prove that equation $z^5 = \frac{1}{10}z^{10} + \frac{1}{15}z^{15}$ has exactly 5 solutions in the unit disk. Estimate the absolute value of the solution with the smallest non-zero modulus.
(c) Evaluate $\int_{\gamma} \frac{z^2 + 1}{z(z^2 + 4)}dz$, $\gamma(t) = re^{2\pi it}, 0 \leq t \leq 1$, for (i) $r = 1$ and (ii) $r = 3$.

Problem 5:
(a) Let $\{x_1, x_2, x_3, \ldots, x_n\}$ and $\{y_1, y_2, y_3, \ldots, y_n\}$ be two different bases of a linear space $V$. Show that one can find two vectors $y_i$ and $y_j$ of the second basis such that the collections $\{y_i, y_j, x_3, \ldots, x_n\}$ and $\{x_1, x_2, y_1, y_2, y_3, \ldots, y_n\}\{y_i, y_j\}$ are again two bases of a linear space $V$.
(b) Prove that if $|a_{ii}| > \sum_{k \neq i} |a_{ki}|$ for $i = 1, 2, \ldots, n$, then the matrix $A = (a_{ij})_{1 \leq i, j \leq n}$ is invertible.
(c) Let $V$ be an inner product vector space, and let $y, z \in V$. Define the linear operator $T : V \to V$ by $T(x) = \langle x, y \rangle z$, for all $x \in V$. Show that the adjoint operator $T^*$ exists and find an explicit expression for it.
Problem 6: Consider the measure space \( \{ \mathbb{R}, \mathcal{L}, m \} \), where \( m \) is Lebesgue measure.

(a) State Fatou’s lemma.

(b) Sketch the proof of the following statement: If \( f \in L^1(\mathbb{R}, m) \), then for any \( \varepsilon > 0 \) there exists a continuous function \( g \) vanishing outside a finite interval and such that
\[
\int_{\mathbb{R}} |f - g| \, dm < \varepsilon .
\]
State all the results you are using.

(c) Assume that \( f_n \in L^1(\mathbb{R}, m) \) and \( f_n \to 0 \) almost everywhere, as \( n \to +\infty \).
Prove
\[
\lim_{n \to +\infty} \int_{\mathbb{R}} \sin(f_n(x)) e^{-x^2} \, dm(x) = 0.
\]

Problem 7:

(a) Let \( C^1[0,1] \) be the space of continuously differentiable functions on \([0,1]\) (at the endpoints we assume the existence of one-sided derivatives), with the norm
\[
\|f\|_{C^1} = \sup_{x \in [0,1]} |f(x)| + \sup_{x \in [0,1]} |f'(x)| .
\]
(Do not prove that this is a norm.) Let \( F : C^1[0,1] \to C^1[0,1] \) be defined by
\[
F(f)(x) = \sin(x)f(x) .
\]
Show that \( F \) is well defined and continuous.

(b) Prove that every totally bounded space is separable. Is every separable metric space also totally bounded?

Problem 8: Let \( X = \{ n \in \mathbb{Z} : n \geq 1 \} \) and let \((n, m)\) denote the greatest common divisor of \( n \) and \( m \).

(i) Prove that for any \( n, m, k \in X \) we have \((n, k) \cdot (m, k) \leq k \cdot (n, m)\).

Let \( d(n, m) = \log \frac{n \cdot m}{(n, m)^2} \).

(ii) Prove that \((X, d)\) is a metric space.

(iii) Prove that \((X, d)\) is unbounded and discrete.

(iv) Find the closed ball \( B(6, \log(4)) \) with center 6 and radius \( \log(4) \).

(v) Is the metric \( d \) equivalent to the standard metric \( \rho(n, m) = |n - m|? \)

(vi) Is the identity a continuous map between \((X, d)\) and \((X, \rho)\)?