Problem 1:
(a) Prove that if \( f \) is continuous on a closed interval \([a, b]\), differentiable on the open interval \((a, b)\) and if \( f(a) = f(b) = 0 \), then for any real \( \alpha \) there is a point \( x \in (a, b) \) such that
\[
\alpha f(x) + f'(x) = 0.
\]
(b) Show that the equation \( 3^x + 4^x = 5^x \) has exactly one real root.
(c) Consider the series
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}e^{-nx}}{\sqrt{n + x^2}}.
\]
Prove that it converges uniformly on \([0, +\infty)\). Prove that the series of moduluses converges pointwise on \((0, +\infty)\).

Problem 2:
(a) Let \( A \) be asymmetric matrix with eigenvalues \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \). Show that
\[
\lambda_k = \min_{S \text{ dim}(S) = k} \left( \max_{\|x\|=1} x^TAx \right),
\]
where \( S \) are subspaces of \( \mathbb{R}^n \).
(b) Let \( A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \). Find an orthogonal matrix \( U \) and a diagonal matrix \( D \)
such that \( U^{-1}AU = D \).
(c) Let \( V \) and \( W \) be finite dimensional subspaces of a vector space. Prove that
\[
\dim(V + W) = \dim(V) + \dim(W) - \dim(V \cap W).
\]
(Hint: start with a basis of \( V \cap W \).

Problem 3: Let \( f : X \to Y \) be a function. Are the following statements true or false? Give a sketch of a proof or a counterexample.
(a) If for any close set \( F \subset X \) the set \( f(F) \subset Y \) is also closed, then \( f \) is continuous.
(b) If \( f \) is continuous and \( F \subset X \) is closed, then \( f(F) \subset Y \) is also closed.
(c) If \( X \) is compact and \( f \) is continuous, then for any closed \( F \subset X \) the image \( f(F) \subset Y \) is also closed.
(d) If \( f \) is continuous and \( G \subset X \) is nowhere dense, then \( f(G) \subset Y \) is also nowhere dense.
**Problem 4:**

(a) Show that the equation \(5z^n = e^z\), \(n \geq 1\),

has no solutions in the annulus \(1 < |z| < 2\). Show that it has at most a finite number of solutions in any horizontal strip \(a < \Re z < b\) and in any vertical strip \(a < \Im z < b\).

(b) Let

\[ f(z) = \frac{1}{(1 + z^2)^2}. \]

(i) What is the radius of convergence of the Taylor expansion of \(f\) centered at \(z_0 = 0\)? You do not have to produce the expansion.

(ii) Expand \(f\) into Laurent series centered at \(z_0 = i\) (a few terms). In what domain is this expansion valid?

(iii) Use the residue theorem to evaluate

\[ \int_{-\infty}^{+\infty} \frac{1}{(1 + x^2)^2} \, dx. \]

**Problem 5:**

(a) Real functions \(f_1, f_2\) are defined on interval \((a, b)\). For any real constants \(c_1, c_2\) the function \(c_1 f_1 + c_2 f_2\) is of constant sign. Prove that \(f_1, f_2\) are linearly dependent.

(b) How many automorphisms there are

(i) from \(\mathbb{Q}\) to \(\mathbb{Q}\) (field of rational numbers);

(ii) from \(\mathbb{R}\) to \(\mathbb{R}\) (field of real numbers);

(iii) from \(\mathbb{C}\) to \(\mathbb{C}\) (field of complex numbers).

(c) Are the following statements true or false? Give a sketch of a proof or a counterexample.

(i) A compact subspace of the reals is closed.

(ii) A closed subspace of the reals is compact.

(iii) A closed subspace of a compact metric space is compact.
Problem 6:
(a) Prove that the only solution to the system
\[
\begin{align*}
\frac{1}{2}x_1 &= a_{11}x_1 + \cdots + a_{1n}x_n ; \\
\frac{1}{2}x_2 &= a_{21}x_1 + \cdots + a_{2n}x_n ; \\
&\vdots \\
\frac{1}{2}x_n &= a_{n1}x_1 + \cdots + a_{nn}x_n ;
\end{align*}
\]
with integer coefficients \(\{a_{ij}\}_{1 \leq i, j \leq n}\) is \(x_1 = x_2 = \cdots = x_n = 0\).
(b) The elements \(u_1, u_2, v_1, v_2\) of a group \(G\) satisfy identities
\[
u_1v_1 = v_1u_1 = u_2v_2 = v_2u_2 ,
\]
and
\[
u_1^{p_1} = u_2^{p_1} = v_1^{p_2} = v_2^{p_2} = e ,
\]
where \(p_1, p_2\) are relatively prime positive integers. Prove that
\[
u_1 = u_2 \quad \text{and} \quad v_1 = v_2 .
\]
(d) Is the following statement true or false? Give a sketch of a proof or a counterexample.
Two matrices with the same minimal polynomial and the same characteristic polynomial are similar.

Problem 7: Consider the measure space \(\{\mathbb{R}, \mathcal{L}, m\}\). Let \(\{f_n\}_{n=1}^\infty\) be a sequence of positive measurable functions.
(a) State Fatou’s lemma.
(b) Prove that if \(f_n \leq f_{n+1}\) for all \(n \geq 1\) and \(f_n \to f\) a.e., then \(\int f_n dm \to \int f dm\).
(c) Prove that
\[
\int \sum_{n=1}^\infty f_n dm = \lim_{k \to \infty} \int \sum_{n=1}^k f_n dm .
\]
(d) State Lebesgue dominated convergence theorem.
(e) Assume that \(f_n \in L^1([0, 1])\) and \(f_n \to 0\) almost everywhere, as \(n \to +\infty\).
Prove
\[
\lim_{n \to +\infty} \int_0^1 \sin(f_n(x)) dx = 0 .
\]
Is the same true if we exchange \([0, 1]\) for the whole line \(\mathbb{R}\)?
Problem 8: Let $C[0, 1]$ be the space of all continuous functions on $[0, 1]$ and let $M[0, 1]$ be the space of all probability measures on $[0, 1]$. Let $\{f_1, f_2, f_3, \ldots\}_{n=1}^{\infty}$ be a set of functions dense in the unit ball $\{f \in C[0, 1] : \sup_x |f(x)| \leq 1\}$ of $C[0, 1]$. (Why such a countable set exists?) For any $\mu_1, \mu_2 \in M[0, 1]$ let us define

$$d(\mu_1, \mu_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} |\mu_1(f_n) - \mu_2(f_n)|,$$

where $\mu_i(f) = \int_0^1 f(x) d\mu_i(x)$.

(a) Prove that $d(\cdot, \cdot)$ is a metric on $M[0, 1]$. (You may need the information that if $\mu(f) = 0$ for all $f \in C[0, 1]$, then $\mu = 0$. You do not have to prove this.)

(b) Prove that the convergence in metric $d$ is equivalent to the vague (weak) convergence of measures:

$$d(\mu_n, \mu) \longrightarrow 0 \iff \forall f \in C[0, 1] \quad \mu_n(f) \longrightarrow \mu(f).$$

(d) Let us define measures $\mu_n$ by

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta(i/n), \quad n = 2, 3, 4, \ldots,$$

where $\delta(x)$ is Dirac’s measure at $x$, i.e.,

$$\delta(x)(\{x\}) = 1 \quad \text{and} \quad \delta(x)([0, 1] \setminus \{x\}) = 0.$$

Show that $\mu_n$ converge weakly to Lebesgue measure, as $n \to \infty$.

Read the problems carefully!
Show All Work!
Justify All Answers!
Good Luck!!!